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VITAL STATISTICS.

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VITAL STATISTICS.

A. Tables of Prussian Mortality, interpolated for Annual Intervals of Age; accompanied with Formulæ and Process for Construction.
B. Discussion of Certain Methods for converting Ratios of Deaths to Population, within given Intervals of Age, into the Logarithm of the Probability that one living at the Earlier Age will attain the Later; with Illustrations from English and Prussian Data.
C. Process for deducing accurate Average Duration of Life, present Value of Life-Annuities, and other useful Tables involving Life-Contingencies, from Returns of Population and Deaths, without the Intervention of a General Interpolation.

THE mortality and accompanying tables, to which the attention of the Association is called, comprise portions of a series of tables that have been and are being prepared, for the New England Mutual Life Insurance Company of Boston, from official returns of the British, Swedish, Prussian, and Belgian governments, and from such reliable American statistics as are obtainable.

In several of the United States of America the decennial enumeration of the numbers and ages of the living effected for the General Government have been quite accurate and reliable, while the only official *mortality* returns (viz. those ordered in connection with the last census, 1850) are inaccurate and deficient. In Massachusetts, since its Registration Act of 1849, certain districts have furnished valuable and satisfactory information respecting the numbers and ages of the dying; but from the published abstracts it has been impossible to separate imperfect from reliable data. In the yet unpublished abstracts of the returns for 1855 an improvement is being effected, under the direction of the present Secretary of State, which, although augmenting somewhat the expense, will afford fit material for the construction of a Life-Table that shall satisfactorily represent the rates of mortality prevailing among the inhabitants of the larger part of the Commonwealth.

The leading paper (A) presents a new Life-Table, complete for annual intervals of age, and calculated from over a million (1,197,407)

With the Respects of the Author.

of observations regarding the ages of the dying, in a population of fifteen millions (14,928,501), and in a community where observations on vital statistics, for many years, are believed to have been made with care and accuracy. It adds *one* to the very limited list of National Life-Tables.

The remaining papers (B and C) are devoted to the discussion of certain methods for converting rates of mortality for different intervals of age into probability of living; and to the presentation of *abridged methods* for calculating, at certain ages, accurate tables of practical value, involving life-contingencies, accompanied with simple rules for determining any required value intermediate.

A. Tables of Prussian Mortality, interpolated for Annual Intervals of Age; accompanied with Formulæ and Process for Construction.

The data from which the following tables have been calculated were obtained from documents sent by Mr. Hoffman of Berlin to the English Ministry of Foreign Affairs, and published in the Sixth Annual Report of the Registrar-General in England.

Population of Prussia, Civil and Military (exclusive of Neufchatel).*

At the end of the year 1834, 13,509,927.
" " 1837, 14,098,125.
" " 1840, 14,928,501.

The documents above mentioned give no statistics of immigration or emigration.

The increase of population during the three years 1838, '39, '40, was 830,376.

The excess of births over deaths during the same three years was 486,937.

Leaving 343,439, which is 41.36 per cent of the total increase of population, unaccounted for by excess of births over deaths.

* "The population of Neufchatel, not included in the above, was 59,448 in 1837, 52,223 in 1825."

Population of Prussia at the End of the Year 1840, classed according to Age and Sex.

Ages.	Males.	Females.	Males.	Females.	Ages.
0 - 5	1,134,413	1,114,871			
5 - 7	370,740	336,429	2,603,699	2,550,022	0 - 14
7 - 14	1,098,546	1,068,722			
14 - 16	344,179	331,039			14 - 16
16 - 20	586,059				
20 - 25	692,704				
25 - 32	777,183		3,238,434	3,253,643	16 - 45
32 - 39	646,122				
39 - 45	536,366				
45 - 60	816,726	881,280			45 - 60
60 and upwards,	445,544	463,935			60 and upwards.
All Ages,	7,448,582	7,479,919			

Assuming the distribution of the (3,253,643) females for the several intervals between the ages 16 and 45 to be proportioned to the distribution of (3,238,434) the corresponding number of males, we have

Ages.	Females.
16 - 20	588,812
20 - 25	695,957
25 - 32	780,833
32 - 39	649,156
39 - 45	538,885
Total, 16 - 45	3,253,643

Hence the following

Numbers and Ages of the Population of Prussia at the End of the Year 1840.

Ages.	Persons.
0 - 5	2,249.284
5 - 7	737,169
7 - 14	2,167,268
14 - 16	675,218
16 - 20	1,174,871
20 - 25	1,388,661
25 - 32	1,558,016
32 - 39	1,295.278
39 - 45	1,075,251
45 - 60	1,698,006
60 and upwards,	909,479
All Ages,	14,928,501

We wish to distribute the population from ages 25 to 45, from 45 to

60, and from 60 upwards in quinquennial or decennial periods, to correspond with the ages of the dying as presented in the mortality returns.

We first determine the quinquennial distribution between ages 25 and 45.

Let $P_{x/y}$ represent the population between ages x and y, or the numbers living under age y, *less* the numbers living under age x.

Then
$$P_{16/20} = 1{,}174{,}871$$
$$P_{16/25} = 2{,}563{,}532$$
$$P_{16/32} = 4{,}121{,}548$$
$$P_{16/39} = 5{,}416{,}826$$
$$P_{16/45} = 6{,}492{,}077$$
$$P_{16/60} = 8{,}190{,}083$$

Assume
$$P_{16/x} = P_{16/a}\frac{x-b\,.\,x-c}{a-b\,.\,a-c} + P_{16/b}\frac{x-a\,.\,x-c}{b-a\,.\,b-c} + P_{16/c}\frac{x-a\,.\,x-b}{c-a\,.\,c-b}$$

Let $a = 20$, $b = 25$, $c = 32$; then will $P_{16/30} = 3{,}722{,}366$.

Let $a = 25$, $b = 32$, $c = 39$; then will $P_{16/30} = 3{,}703{,}211$, and $P_{16/35} = 4{,}708{,}839$.

Let $a = 32$, $b = 39$, $c = 45$; then will $P_{16/35} = 4{,}682{,}050$, and $P_{16/40} = 5{,}598{,}277$.

Let $a = 39$, $b = 45$, $c = 60$; then will $P_{16/40} = 5{,}611{,}751$.

Taking the arithmetical mean of the above duplicate results, we have

$$P_{16/30} = 3{,}712{,}789$$
$$P_{16/35} = 4{,}695{,}445$$
$$P_{16/40} = 5{,}605{,}014$$

Hence
$$P_{25/30} = 1{,}149{,}257$$
$$P_{30/35} = 982{,}656$$
$$P_{35/40} = 909{,}569$$
$$P_{40/45} = 887{,}063$$

which results cannot vary materially from the actual distribution.

The following Table gives the distribution of the population of Prussia between ages 45 and 60 (1,698,006); and of the population from age 60 upwards, according to the corresponding proportional distribution of the numbers of the population of the Northwestern Division of England (the Eighth of the eleven Districts into which England and Wales are divided in the Reports of the Registrar-General).

Ages.	Distribution of Prussian Population over age 45.
45 – 55	1,257,322
55 – 60	440,684
60 – 65	353,657
65 – 75	398,925
75 – 85	137,188
85 – 95	18,638
95 and over,	1,026
45 and over,	2,607,485

NUMBERS AND AGES OF THE POPULATION OF THE NORTHWESTERN DIVISION (ENG.) IN 1841, ACCORDING TO WHICH THE ABOVE DISTRIBUTION WAS MADE. — (9th Rep. Reg.-Gen.)

Ages.	Northwestern Division (Eng.), 1841. Numbers and Ages of the Living above Age 45.
45 - 55	151,064
55 - 60	52,947
60 - 65	39,656
65 - 75	44,732
75 - 85	15,383
85 - 95	2,095
95 and over,	115
45 and over,	305,992

The numbers living above age 15 in the Northwestern Division (England) were grouped, with reference to age, only in decennial classes.

By assuming the algebraic equation,

$$P_{55/x} = P_{55/45} \cdot \frac{x-55 \,.\, x-65 \,.\, x-75}{45-55 \,.\, 45-65 \,.\, 45-75} + P_{55/65} \cdot \frac{x-45 \,.\, x-55 \,.\, x-75}{65-45 \,.\, 65-55 \,.\, 65-75} + P_{55/75} \cdot \frac{x-45 \,.\, x-55 \,.\, x-65}{75-45 \,.\, 75-55 \,.\, 75-65}$$

a close approximation to the probable number of persons living between ages 55 and 60 (52,947) and between ages 60 and 65 (39,656) resulted.

$P_{55/x}$ represents the number of persons reported living under age x, *less* the number living under age 55.

We remark that $P_{55/45}$ is essentially negative.

The population of the Division, as returned for the night of June 6 - 7, 1841, was two millions (2,098,820), being one eighth of the entire population of England and Wales (15,914,148) at that date. The counties of Cheshire and Lancashire constitute this Division. The latter county includes the densely populous and unhealthy district of Liverpool. [illegible]

The ratios of deaths to population for the intervals from age 45 to 60, and from age 60 upwards, more closely approximated the corresponding ratios for Prussia, than did those of any other large community concerning which reliable population and mortality statistics were to be obtained.

TABLE COMPARING RATIOS OF THE ANNUAL NUMBER OF DEATHS TO THE NUMBERS LIVING IN CERTAIN COMMUNITIES FROM AGE 45 TO 60, AND FROM AGE 60 TO EXTREME OLD AGE.

	Ages 45 - 60.	Ages 60 and upwards.
Prussia.		
Deaths, 1839, '40, '41, / Population, 1840, ...	.024	.088
Northwestern Division (England).		
Deaths, seven years, 1838 - 44, / Population, middle, 1841, . .	.023	.079+
Sweden.		
Deaths, twenty years, 1821 - 40, / Population, mean of 1820, '30, '40, . .	.021	.079—
Belgium.		
Deaths, nine years, 1842 - 50, / Population, October 15, 1856, . . .	.020	.073
England and Wales.		
1841, ...	.019	.069

A comparison of the distribution of the numbers of the living in Prussia in these intervals of age according to that of the Northwestern Division (Eng.), with a distribution of the same numbers according to the mean of the corresponding distribution of equal numbers of the populations of England in 1841 and of Belgium in 1846, would give the following results.

DISTRIBUTION OF THE POPULATION OF PRUSSIA ACCORDING TO

Ages.	The Mean of Equal Numbers in England and Belgium.	The Northwestern Division (England).
45 - 55	1,285,567	1,257,322
55 - 60	412,439	440,684
45 - 60	1,698,006	1,698,006
60 - 65	330,425	353,657
65 - 75	398,646	398,925
75 - 85	155,077	137,188
85 and over,	25,331	19,709
60 and over,	909,479	909,479
45 and over,	2,607,485	2,607,485

The distribution according to the English and Belgian facts would give larger numbers after about age 75, in the resulting Life-Table.

The distribution according to that of the Northwestern Division was adopted as the best representation of the probable corresponding distribution of the population of Prussia, within the intervals of age above mentioned. Hence the following Table.

Deaths, Population, Mortality, and Logarithms of the Probability of Living in Prussia.

The Numbers of the Living between ages 45 *and* 60, *and from* 60 *to extreme old age are distributed according to corresponding proportional distributions of the numbers of the population of the Counties of Cheshire and Lancashire* (*Northwestern Division*), *in England, in* 1841.

Ages. x, y.	Deaths.	Population.	Mortality.	Logarithms of Probabilities of Surviving each Interval.	
	Aggregate Numbers and Ages of the Dying during the Three Years 1839, '40, '41.	Numbers and Ages of the Living at the End of the Year 1840.	Ratios of the Average Annual Numbers of the Dying during the Three Years 1839, '40, '41, to the Numbers of the Living computed with reference to the Middle of 1840.	Duplicate Values, each deduced from two consecutive Ratios in the Column of Mortality.	Values derived from Comparison of the Duplicates.
	$D_{0/y} - D_{0/x}$	$P_{0/y} - P_{0/x}$	$\frac{D_{x/y}}{P_{x/y}} \cdot C$	$\lambda p_{0/y} - \lambda p_{0/x}$	
	$D_{x/y}$	$P_{x/y}$	$M_{x/y}$	$\lambda p_{x/y}$	
0 – 1	310,527				
1 – 3	162,356	2,249,284	.0802238		
3 – 5	62,734				
5 – 7	33,272	737,169	.0152056	—.013106 —.013201	—.013155
7 – 14	27,156	2,167,268	.0077790	—.023480 —.023628	—.023557
14 – 20	22,887	1,850,089	.0062978	—.016399 —.016432	—.016416
20 – 25	34,585	1,388,661	.0089397	—.019433 —.019418	—.019425
25 – 30	36,849	1,149,257	.0096939	—.021057 —.021059	—.021058
30 – 35	31,594	982,656	.0108317	—.023533 —.023542	—.023537
35 – 40	35,579	909,569	.0131780	—.028646 —.028630	—.028637
40 – 45	38,094	887,063	.0144675	—.031434 —.031464	—.031449
45 – 55	78,503	1,257,322	.0210345	—.092155 —.092527	—.092322
55 – 60	46,704	440,684	.0357042	—.077947 —.078021	—.077981
60 – 65	58,576	353,657	.0557995	—.122543 —.121891	—.122189
65 – 75	107,653	398,925	.0909134	—.424020 —.408584	—.415608
75 – 85	61,697	137,188	.1515098	—.716433 —.730677	—.722021
85 and upwards	15,572	19,709	.2661784		
Total,	1,197,407	14,928,501			

$$C = \tfrac{1}{3} \cdot \frac{14928501}{14770727}$$

14928501 = Population of Prussia, as returned for the end of the year 1840.

14770727 = Population of Prussia, estimated for the middle of the year 1840, from the numbers returned as living at the end of 1834, 1837, and 1840.

It will be observed that the values derived from comparison of the duplicate logarithms, and which have been adopted in constructing the Interpolated and other Tables, are not in all cases arithmetical means. The difference is of little moment, but there is no sufficient reason for preferring the former.

LOGARITHMS OF THE PROBABILITY OF SURVIVING,

Computed from the Returns of the Numbers of the Living under Age 5; *and of the Numbers of the Dying annually under* 1 *Year of Age, over* 1 *and under* 3, *over* 3 *and under* 5.

Ages.	$\lambda\, p_{x/y}$.
0 – 1	— .082920
1 – 3	— .051670
3 – 5	— .022522

The successive addition of the logarithms of the probabilities of surviving the consecutive intervals of age to 5.001688, the logarithm assumed for the proportional numbers born alive, gives the following

TABLE OF THE LOGARITHMS OF THE PROPORTIONS, AND THE PROPORTIONS OF PERSONS BORN ALIVE AND SURVIVING CERTAIN AGES IN PRUSSIA, ACCORDING TO THE CALCULATED LAW OF MORTALITY.

Deaths, 1839, '40, '41.

Population computed with reference to middle of 1840.

Distribution of Population above Age 45, *Northwestern Division* (*Eng.*).

Age.	Survivors.	
	Logarithms.	Persons.
	$\lambda\, L_x$	L_x
0	5.001688	100,389
1	4.918768	82,941
3	4.867098	73,637
5	4.844576	69,916
14	4.807864	64,249
25	4.772023	59,159
35	4.727428	53,386
45	4.667342	46,488
55	4.575020	37,585
65	4.374850	23,706
75	3.959242	9,104.2
85	3.237221	1,726.7
95	1.982879	96.1
105	— 1.803755	.636

The values opposite ages 95 and 105 were computed from the logarithms of the numbers surviving at ages 65, 75, and 85, by the exponential formula,

$$\lambda L_x = \Phi_x = \Phi_{65} + (\Phi_{75} - \Phi_{65}) \frac{q^{x-65} - 1}{q^{75-65} - 1};$$

in which

$$q^{75-65} \text{ (or } q^{10}) = \frac{\Phi_{85} - \Phi_{75}}{\Phi_{75} - \Phi_{65}}.$$

These values were adopted as bases for the construction of the accompanying Life-Table *interpolated* for annual intervals of age; and also for computing by *abridged methods* certain practical life-contingency tables.

Before presenting this table and these methods, we will state some of the principles which underlie, and indicate the process by which ratios of the numbers of the dying to the numbers of the living, during the several intervals of age, have been converted into logarithms of the probabilities that one living at the earlier age will attain the later.

Whenever, in any community, the intensity of mortality at each age, or the ratio of the numbers momentarily dying during each *minute* interval of age to the numbers then living within the same interval, has been *constant* for a period of time equal to the difference between the specified age and the extreme of old age, an *invariable law of mortality* is said to prevail in that community.

The law of human mortality is seldom strictly invariable. It fluctuates within *certain limits*, not only with different communities and localities, but in the same community during successive periods, and in the same localities. The habits, occupations, and social condition of the members of the community remaining unchanged, the larger their numbers the *narrower* these limits. It is within the province of the vital statistician to determine, not merely an average of the rates of mortality prevailing in a community, but also the sensible limits within which the rates fluctuate.

Our present inquiries have reference to the determination of a law of mortality which shall satisfactorily represent the average of the rates prevailing among the inhabitants of a populous state, with fixed geographical boundaries; and in which the numbers of the inhabitants vary with births and with deaths, with immigration and with emigration.

If, in a large community, varying with births, deaths, and migrations, but in which the numbers of the population have not been subject to sudden and irregular change, the number of the dying during a given year or period of time between ages not very remote be divided by the number of the living between the same ages at the middle of that period of time, the quotient resulting from the division has generally been assumed closely to approximate the quotient that would have resulted had the numbers of the population within the limits of these ages been *stationary;* that is, assuming an invariable law of mortality, had the numbers of the population for each minute interval of age within the limits of these ages remained constant during that period, and unaffected by either immigration or emigration.

The errors involved in this assumption are of small moment compared with probable errors of observation, and vanish when the intervals of age are taken exceedingly minute, and where the excess or deficiency of the deaths in the former half of the period of time, with reference to half the deaths of the entire period, is exactly counterbalanced by a corresponding deficiency or excess of the deaths in the latter half of that period of time.

We adopt this hypothesis, and assume that each of the ratios in the column headed Mortality is identical with that which would have resulted had the population of Prussia within the limits of the ages been stationary for a period of years equal to the specified interval; and we also assume the accuracy of the Prussian mortality and population returns.

From these ratios we now proceed to determine duplicate logarithms of the probability that one surviving the earlier age in each interval will attain the later.

Let $P_{0,x}$ = the number living under age x, in a stationary population, in which the same law of mortality prevails as in Prussia.

$D_{0,x}$ = the number of annual deaths under age x in the stationary population.

l_0 = the number born alive each *moment of time*, in the stationary population.

l_x = the number surviving x years, out of (l_0) the momentary number of births.

$p_{a/b} = \frac{l_b}{l_a}$ = the probability that one surviving the earlier age (a) will attain the later (b).

$\delta_{0/x} = l_0 - l_x$ = the number dying in x years out of (l_0) the momentary number of births.

In a stationary population

$$D_{0/x} = \frac{\delta_{0/x}}{d\,x},$$

and

$$d\,P_{0/x} = l_x\,;$$

therefore,

$$P_{0/x} = \int (l_0 - \delta_{0/x}) = \frac{x\,.\,l_0}{d\,x} - \int \delta_{0/x}\,.$$

But $M_{a/b}$, the ratio of the average annual number of deaths in Prussia between ages a and b to the number of the living between these ages, computed with reference to the middle of the period in which the deaths occur, equals $\frac{D_{a/b} = D_{0/b} - D_{0/a}}{P_{a/b} = P_{0/b} - P_{0/a}}$.

Assume $\delta_{0/x} = Q\,x + R\,x^2$, Q and R being unknown, and independent of the variable x.

Then

$$D_{0/x} = \frac{Q\,x + R\,x^2}{d\,x},$$

and

$$\int \delta_{0/x} = \left(Q\frac{x^2}{2} + R\frac{x^3}{3}\right)\frac{1}{d\,x}.$$

Hence

$$M_{a/b}\left(\text{which} = \frac{D_{a/b}}{P_{a/b}} = \frac{D_{0/b} - D_{0/a}}{P_{0/b} - P_{0/a}}\right) = \frac{Q\,\overline{b-a} + R\,\overline{b^2-a^2}}{l_0\,\overline{b-a} - Q\frac{b^2-a^2}{2} - R\frac{b^3-a^3}{3}}$$

$$= \frac{Q + R\,\overline{b+a}}{l_0 - Q\frac{b+a}{2} - R\frac{b^2+a\,b+a^2}{3}},$$

and

$$M_{b/c} = \frac{Q + R\,\overline{c+b}}{l_0 - Q\frac{c+b}{2} - R\frac{c^2+b\,c+b^2}{3}}.$$

So also,

$$\lambda p_{a/b} \text{ (which } = \lambda \frac{l_b}{l_a} = \lambda \frac{l_0 - \delta_{0/b}}{l_0 - \delta_{0/a}}\Big) = \lambda \frac{l_0 - Q\,b - R\,b^2}{l_0 - Q\,a - R\,a^2},$$

and

$$\lambda\, p_{b/c} = \lambda \frac{l_0 - Q\,c - R\,c^2}{l_0 - Q\,b - R\,b^2}.$$

Given $M_{a/b}$ and $M_{b/c}$, required $\lambda\, p_{a/b}$ and $\lambda\, p_{b/c}$.

First determine values for Q and R; the values of $\lambda\, p_{a/b}$ and $\lambda\, p_{b/c}$ are then readily found.

$$Q = \frac{\gamma' M_{a/b} - \beta' M_{b/c}}{\gamma' \beta - \beta' \gamma} \cdot l_0,$$

and

$$R = \frac{\gamma M_{a/b} - \beta M_{b/c}}{\gamma \beta' - \beta \gamma'} \cdot l_0;$$

in which

$$\beta = 1 + \frac{(b+a)\, M_{a/b}}{2};$$

$$\beta' = b + a + \frac{(b^2 + b\,a + a^2)\, M_{a/b}}{3};$$

$$\gamma = 1 + \frac{(c+b)\, M_{b/c}}{2};$$

$$\gamma' = c + b + \frac{(c^2 + c\,b + b^2)\, M_{b/c}}{3}.$$

The reduction may be simplified by letting $l_0 = 1$, and by the use of addition and subtraction logarithms.

In like manner, from $M_{b/c}$ and $M_{c/d}$ obtain $\lambda\, p_{b/c}$ and $\lambda\, p_{c/d}$; and so on for all intervals of age which the returns give.

We thus obtain *duplicate* values for the logarithms of the probabilities of surviving all the intervals specified except two. For the first and the last interval we have but single values. We may, without material error, adopt for the true probability the mean of these duplicate results.

It will be observed that the conversion, in each of these cases, is made for the *entire* interval, not, as is more frequent, for the *middle*

year of the interval. We are thus enabled, without the intervention of a general interpolation, to compute directly the number surviving at certain ages in the resulting life-table, out of a specified number born alive.

Usually the conversion is from a *single* ratio, based upon the assumption of a uniform distribution of deaths throughout the interval. By the present method, however, the conversion is effected, taking into account the actual or variable distribution of deaths, from *three* consecutive ratios, one preceding and another following the interval. A comparison of the relative accuracy and simplicity of several methods for effecting the conversion will be given on a following page.

We now proceed to indicate methods for obtaining *probabilities of surviving from birth to ages one, three, and five.*

We have the average annual number of deaths in Prussia under the ages of one, three, and five ($D_{0/1}$, $D_{0/3}$, $D_{0/5}$), for the period of the three years 1839, '40, '41; and the population under the age of five ($P_{0/5}$) at the end of the middle year of the period (end of 1840); also the *ratio* $\left(\frac{1}{v}\right)$ of the annual increase in the number of births deduced from the numbers registered for each of the six years 1836 – 41. The average annual number of deaths for the three years 1839, '40, '41 we shall consider identical with the number of deaths for the year 1840.

From the following, it would appear that the *accurate* number of those born alive cannot be obtained directly from official reports, because of probable deficiencies in registration. If the numbers of the living and of the dying at the earlier ages have been accurately observed and returned, if the numbers at these ages have been but little affected by immigration and emigration, and if the ratio of annual increase in the number of births can be obtained, a close approximation to the actual number of those born alive may be computed.

Let l_0 be the number of those momentarily born alive in Prussia at the time for which the census was taken (end of 1840).

$\frac{1}{v} = \left(\frac{\text{births 1839, '40, '41}}{\text{births 1836, '37, '38}}\right)^{\frac{1}{3}} = 1.01545$, the ratio of the annual increase in the number of births estimated from those registered for each of the six years 1836 – 41.

$\lambda \frac{1}{v} = .0066586$, the logarithm of this ratio.

Let $\delta_{0/x}$ be the number that died before attaining the age of x years (according to the prevailing law of mortality) ou of (l_0) the number born alive in Prussia during the moment of time (end of 1840) that the enumeration of the living is supposed to have been made.

$v^x \, d\, \delta_{0/x}$ will express the number of those aged x years that died in Prussia during the supposed moment of enumeration.

$$\int_0^x v^x \, d\, \delta_{0/x} \int_0^1 v^x = D_{0/x},$$

the annual number of deaths in Prussia under the age of x years, for the year ending with the census, i. e. for the year 1840.

$$v^x \int_0^x v^{-x} \int_0^x v^x \, d\, \delta_{0/x}$$

represents the total number that died in Prussia during the x years preceding the time of the enumeration of the living, out of the numbers born alive within that period. This expression obviously equals

$$v^x \int_0^x v^{-x} \frac{D_{0/x}}{\int_0^1 v^x} \,.$$

$$l_0 \int_0^x v^x$$

represents the number born alive during the x years preceding the time of the enumeration.

The numbers born alive within this period of x years, *less* the numbers dying within the period out of the numbers born alive, obviously represent the numbers of the living at the end of the period under the age of x years; immigration and emigration among those under age x being considered null.

$$P_{0/x} = l_0 \int_0^x v^x - v^x \int_0^x v^{-x} \int_0^x v^x \, d\, \delta_{0/x}$$

$$= l_0 \int_0^x v^x - v^x \int_0^x v^{-x} \frac{D_{0/x}}{\int_0^1 v^x}.$$

$$l_0 \int_0^1 v^x = L_0,$$

the numbers born alive during (1840) the year immediately preceding the time of enumeration.

Hence

$$L_0 \left(= l_0 \int_0^1 v^x\right) = \left\{P_{0/x} + v^x \int_0^x v^{-x} \int_0^x v^x \, d\, \delta_{0/x}\right\} \frac{\int_0^1 v^x}{\int_0^x v^x}$$

$$= \left\{P_{0/x} + v^x \int_0^x v^{-x} \frac{D_{0/x}}{\int_0^1 v^x}\right\} \frac{\int_0^1 v^x}{\int_0^x v^x}.$$

$$d\, x \int_0^x v^x = \frac{v^x - 1}{V};$$

in which V is the Napierian logarithm of v.

Let $x = 5$; then will

$$L_0 = \left\{P_{0/5} + \int_0^5 \frac{v^5\, V}{v-1} \cdot \frac{D_{0/x}\, d\, x}{v^x}\right\} \frac{v-1}{v^5 - 1}.$$

To simplify, let

$$D'_{0/x}\, d\, x = \frac{v^{5-x}\, D_{0/x}}{\int_0^1 v^x} = \frac{v^5\, V}{v-1} \cdot \frac{D_{0/x}\, d\, x}{v^x}.$$

Then

$$L_0 = \{P_{0/x} + \int_0^5 D'_{0/x}\, d\, x\} \frac{v-1}{v^5 - 1}.$$

The returns give

$$\left.\begin{array}{l} D_{0/1} = \quad 103{,}509 \\ D_{0/3} = \quad 157{,}628 \\ D_{0/5} = \quad 178{,}539 \end{array}\right\}$$ average annual deaths under ages one, three, and five.

$P_{0/5} = 2{,}249{,}284$ $\Big\{$ population under age five at the end of the year 1840.

From these, and from .0066586 $\left(= \lambda \frac{1}{v}\right)$ the logarithm of the ratio of annual increase among registered births, we find

$$D'_{0/1} = \quad 98{,}100,$$
$$D'_{0/3} = 154{,}043,$$
$$D'_{0/5} = 179{,}912.$$

Assume

$$D'_{0/x} = D'_{0/0}\,[= 0] + x\,\theta + x\,.\,\overline{x-1}\,\theta^2 + x\,.\,\overline{x-1}\,.\,\overline{x-3}\,\theta^3 + x\,.\,\overline{x-1}\,.\,\overline{x-3}\,.\,\overline{x-5}\,R$$

$$= x\,\theta + (x^2 - x)\,\theta^2 + (x^3 - 4\,x^2 + 3\,x)\,\theta^3 + (x^4 - 9\,x^3 + 23\,x^2 - 15\,x)\,R.$$

$$\int_0^x D'_{0/x}\,dx = \frac{x^2}{2}\theta + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\theta^2 + \left(\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right)\theta^3$$

$$+ \left(\frac{x^5}{5} - \frac{9x^4}{4} + \frac{23x^3}{3} - \frac{15x^2}{2}\right) R$$

$$= \frac{x^2}{2}\left\{\theta + \frac{2x-3}{3}\theta^2 + \frac{(3x-16)x+18}{6}\theta^3\right.$$

$$\left. + \frac{[(12x-135)x+460]x-450}{30} R\right\}.$$

$$\frac{d\,D'_{0/x}}{dx} = \theta + (2x-1)\theta^2 + (3x^2 - 8x + 3)\theta^3$$

$$+ (4x^3 - 27x^2 + 46x - 15) R.$$

$$\frac{d^2\,D'_{0/x}}{(dx)^2} = 2\theta^2 + (6x-8)\theta^3 + (12x^2 - 54x + 46) R.$$

θ, θ^2, and θ^3 are the *divided differences* of the values $D_{0/0}$ ($= 0$), $D_{0/1}$, $D_{0/3}$, and $D_{0/5}$; and R is indeterminate.

	Δ	θ	$\Delta\theta$	θ^2	$\Delta\theta^2$	θ^3
$D'_{0/0} =$ 0 000						
	98,100	98,100.				
$D'_{0/1} =$ 98,100			— 70,128.5	— 23,376.17		
	55,943	27,971.5			19,616.92	3,923.38
$D'_{0/3} =$ 154,043			— 15,037.0	— 3,759.25		
	25,869	12,934.5				
$D'_{0/5} =$ 179,912						

We observe that the divided differences of the first order are *positive*, and that they *diminish* as the age advances.

Required for R a value such that the *first* differential coefficients of the function assumed for $D'_{0/x}$ be *positive*. It would also be desirable, if possible, that the *second* differential coefficients, from birth to at least age five, be *negative*.

The latter is not possible for the entire period, with our present values for $D'_{0/1}$, $D'_{0/3}$, and $D'_{0/5}$, if we assume but one arbitrary value (R). Our object, however, is sufficiently attained by taking, for R, a value such that for ages *three* and *five* the above conditions shall be observed.

That the first differential coefficients be *positive* for ages three and five, it is requisite that

$$R < 396.6$$
$$> -920.1\,;$$

that the second differential coefficients be *negative* for the same ages, it is requisite that

$$R > -939.8$$
$$< -520.6\,;$$

from which it appears that R should be *negative*, and that its value be

between — 920.1
and — 520.6

Let $R = -700$. We now have

$$\begin{aligned} \theta &= 98{,}100 \\ \theta^2 &= -23{,}376.17 \\ \theta^3 &= 3{,}923.38 \\ R &= -700. \end{aligned}$$

Hence

$$\int_0^5 D'_{0/x}\,dx \text{ (which } = \frac{25}{2}\theta + \frac{175}{6}\theta^2 + \frac{325}{12}\theta^3 - \frac{125}{12}R) = 657{,}995.$$

$$L_0 = \left\{P_{0/5} + \int_0^5 D'_{0/x}\,dx\right\}\frac{v-1}{v^5-1}\,;$$

in which

$$\lambda\,\frac{v-1}{v_5-1} = \bar{1}.3142433,$$

and

$$P_{0/5} = 2{,}249{,}284\,;$$

therefore,

$$L_0 = 599{,}418.$$

By the above process the probable number born alive during the year 1840 is found to have been 599,418 instead of 562,394, the average of the numbers *registered* as born alive during each of the three years 1839, '40, '41; thereby indicating an annual deficiency in the registration of 37,024, or about 6.2 per cent of the probable number born.

In the above we have supposed the numbers of the dying and of the living at early ages accurately returned. If either be represented less than truth, the resulting correction would give still larger the probable number of births. Correction for deaths that escape registration, if any, would tend to reduce the probabilities of living.

Having found L_0 (which equals $\frac{v-1}{V}\cdot\frac{l_0}{dx}$), the annual number of births for the year 1840, we next seek values, corresponding to intervals of age 0 – 1, 1 – 3, and 3 – 5, for $D''_{0/x}$ (which equals $\frac{v-1}{V}\cdot\frac{d_{0/x}}{dx}$), the annual number of deaths in a stationary popula-

tion in which L_0 is the annual number of births; or the number that must die in x years, according to the law of mortality prevailing in Prussia, out of L_0, born alive.

$$v^x D'_{0/x} = \frac{v^5 V}{v-1} D_{0/x}$$

$$= \frac{v^5 V}{v-1} \int_0^x v^x \frac{d\,\delta_{0/x}}{dx} \cdot \frac{v-1}{V}$$

$$= \frac{v^5 V}{v-1} \int_0^x v^x\, d\, D''_{0/x}.$$

$$\therefore\ v^x\, d\, D''_{0/x} = \frac{v-1}{v^5 V} \cdot d\,(v^x D'_{0/x}).$$

But

$$d\,(v^x \,.\, D'_{0/x}) = v^x \,.\, d\,D'_{0/x} + D'_{0/x} \,.\, d\,v^x = v^x\,(d\,D'_{0/x} + V D'_{0/x}\,d\,x)\,;$$

$$\therefore\ d\,D''_{0/x} = (d\,D'_{0/x} + V D'_{0/x}\,d\,x)\,\frac{v-1}{v^5\,V}\,.$$

Integrating,

$$D''_{0/x} = \frac{v-1}{v^5\,V} D'_{0/x} + \frac{v-1}{v^5} \int_0^x D'_{0/x}\,d\,x.$$

v, V, $D'_{0/1}$, $D'_{0/3}$, and $D'_{0/5}$ are already known; also θ, θ^2, and θ^3, and R in the expression

$$\int_0^x D'_{0/x}\,d\,x = \frac{x^2}{2}\left\{\theta + \frac{2x-3}{3}\theta^2 + \frac{(3x-16)\,x+18}{6}\theta^3 \right.$$
$$\left. + \frac{[(12\,x-135)\,x+460]\,x-450}{30} R\right\}.$$

Substituting for x values 1, 3, and 5, we have

$$\int_0^1 D'_{0/x}\,d\,x = 55{,}899,$$

$$\int_0^3 D'_{0/x}\,d\,x = 323{,}335,$$

$$\int_0^5 D'_{0/x}\,d\,x = 657{,}995.$$

Therefore,

$$D''_{0/1} = 104{,}184,$$
$$D''_{0/3} = 159{,}735,$$
$$D''_{0/5} = 181{,}955.$$

$\lambda p_{0/x}$ (the logarithm of the probability that one born alive will survive x years) $= \lambda \frac{l_0 - \delta_{0/x}}{l_0} = \lambda \frac{L_0 - D''_{0/x}}{L_0}$.

Therefore,

$$\lambda p_{0/1} = \bar{1}.9170804 = \lambda .82619,$$
$$\lambda p_{0/3} = \bar{1}.8654097 = \lambda .73352,$$
$$\lambda p_{0/5} = \bar{1}.8428880 = \lambda .69645.$$

Hence of 100,000 born alive there will attain the age of

one year 82,619,
three years 73,352,
five years 69,645 ;

or of 100,389 born alive there will attain the age of

one year 82,941,
three years 73,637,
five years 69,916.

The latter results are those adopted in the accompanying interpolated and other tables. These tables, as first constructed, represented the probability of surviving five years from birth to be .69916, computed by a process less rigorous and satisfactory than the one just described. By assuming the same number surviving at age five (69,916) as in the original table, modification of the values for ages greater than five becomes unnecessary.

The logarithms of the numbers surviving certain ages out of 100,389 born alive may be continued for ages greater than five, by successively adding to 4.8445759 (the logarithm of the number surviving age five), the logarithms that have previously been determined for the probabilities of surviving the consecutive intervals.

The table will then be ready, either for a general interpolation of the numbers surviving each anniversary of birth, or for obtaining, by abridged methods, the accurate average duration of life, life annuities, annual premiums, single premiums, and other practical tables involving life contingencies, for certain ages, without the intervention of a general interpolation. Simple rules may also be added for computing from these periodical results any specified values intermediate.

The following is a brief method for finding *approximate* values for the probabilities of surviving the intervals from birth to ages one, three, and five, on the supposition of a probable deficiency in the registered number of births, and that the ratio between the numbers registered and the true numbers is constant.

The same interpretation of symbols is observed as in the last demonstration.

We already have

$$L_0 = l_0 \int_0^1 v^x = \frac{P_{0/5} + \int_0^5 D'_{0/x}\, d\, x}{\int_0^5 v^x} \int_0^1 v^x,$$

$$D'_{0/x}\; d\, x = \frac{v^{5-x}\, D_{0/x}}{\int_0^1 v^x},$$

and

$$D_{0/x} = \int_0^x v^x\, d\, \delta_{0/x} \int_0^1 v^x.$$

When the interval $(0 - x)$ is not large, $x\, v^{\frac{x}{2}}$ is a close approximation to the value of $\int_0^x v^x\, d\, x$; hence the following approximate relations.

$$L_0 = \frac{l_0}{d\, x} v^{\frac{1}{2}} = \left\{ P_{0/5} + \int_0^5 D'_{0/x}\, d\, x \right\} \frac{v^{\frac{1}{2}}}{5\, v^{\frac{5}{2}}}.$$

$$D'_{0/x} = \frac{v^5}{v^{\frac{1}{2}}} \cdot \frac{D_{0/x}}{v^x}.$$

$$v^{\frac{1}{2}} \cdot \frac{d\, \delta_{0/x}}{d\, x} = \frac{d\, D_{0/x}}{v^x}.$$

Let us first seek an approximate value for L_0.

It is obvious that

$$\int_0^5 D'_{0/x}\, d\, x = \int_3^5 D'_{0/x}\, d\, x + \int_1^3 D'_{0/x}\, d\, x + \int_0^1 D'_{0/x}\, d\, x.$$

Assuming each term, in the right-hand member, to be the integral of the general term of an *equidifferent* progression, we have

$$\int_3^5 D'_{0/x}\, d\, x = \overline{5-3}\; \frac{D'_{0/5} + D'_{0/3}}{2}.$$

$$\int_1^3 D'_{0/x}\, d\, x = \overline{3-1}\; \frac{D'_{0/3} + D'_{0/1}}{2}.$$

$$\int_0^1 D'_{0/x}\, d\, x = \overline{1-0}\; \frac{D'_{0/1}}{2}.$$

Therefore,

$$\int_0^5 D'_{0/x}\, d\, x = D'_{0/5} + 2\, D'_{0/3} + \tfrac{3}{2}\, D'_{0/1}.$$

Since

$$D'_{0/x} = \frac{v^5}{v^{\frac{1}{2}}} \cdot \frac{D_{0/x}}{v^x},$$

$D'_{0/1}$, $D'_{0/3}$, and $D'_{0/5}$ equal respectively 98101, 154044, and 179913.

Therefore,

$$\int_0^5 D'_{0/x}\, d\,x = 635{,}152.$$

But

$$L_0 = \frac{P_{0/5} + \int_0^5 D'_{0/x}\, d\,x}{5\,v^2}$$

$$= \frac{2{,}249{,}284 + 635{,}152 = 2{,}884{,}436}{5\,v^2}$$

$$= 594{,}851.$$

594,851, the computed number of births for the year 1840 by this *approximate* method, is *less* by about three fourths of one per cent than 599,418, the corresponding number of births computed by the previous method.

Having found an approximate value for

$$v^{\frac{1}{2}} \cdot \frac{l_0}{d\,x} \text{ (or } L_0\text{)},$$

we next wish approximate values for

$$v^{\frac{1}{2}} \cdot \frac{\delta_{0/x}}{d\,x} \left(\text{or} \int_0^x \frac{d\,D_{0/x}}{v^x}\right),$$

corresponding to intervals of age 0 – 1, 0 – 3, and 0 – 5.

When the interval $b - a$ is small,

$$\int_a^b \frac{d\,D_{0/x}}{v^x} \text{ nearly equals } \frac{D_{0/b} - D_{0/a}}{v^{\frac{b+a}{2}}}, \text{ or } \frac{D_{a/b}}{v^{\frac{b+a}{2}}}.^*$$

Hence the following approximations:

$$v^{\frac{1}{2}}\,\frac{\delta_{0/1}}{d\,x} = \frac{D_{0/1}}{v^{\frac{1}{2}}} = 104{,}306.$$

$$v^{\frac{1}{2}}\,\frac{\delta_{1/3}}{d\,x} = \frac{D_{1/3}}{v^2} = 55{,}804.$$

$$v^{\frac{1}{2}}\,\frac{\delta_{3/5}}{d\,x} = \frac{D_{3/5}}{v^4} = 22{,}234.$$

$$\therefore\ v^{\frac{1}{2}}\,\frac{\delta_{0/1}}{d\,x} = 104{,}306.$$

$$v^{\frac{1}{2}}\,\frac{\delta_{0/3}}{d\,x} = 160{,}110.$$

$$v^{\frac{1}{2}}\,\frac{\delta_{0/5}}{d\,x} = 182{,}344.$$

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* This approximation was adopted by Dr. Farr in constructing his Austrian Life-Table. — *Rep. Reg. Gen.*

Hence

$$p_{0/1}\ \left(\text{which} = \frac{L_0 - \frac{\delta_{0/1}\, v^{\frac{1}{2}}}{d\,x}}{L_0}\right) = \lambda^{-1}\,\bar{1}.9162709 = .82465.$$

$$p_{0/3}\ \left(\text{which} = \frac{L_0 - \frac{\delta_{0/3}\, v^{\frac{1}{2}}}{d\,x}}{L_0}\right) = \lambda^{-1}\,\bar{1}.8638226 = .73084.$$

$$p_{0/5}\ \left(\text{which} = \frac{L_0 - \frac{\delta_{0/5}\, v^{\frac{1}{2}}}{d\,x}}{L_0}\right) = \lambda^{-1}\,\bar{1}.8410233 = .69346.$$

Then of 100,000 born alive there will be living at ages

(1) 82,456,
(3) 73,084,
(5) 69,346;

or assuming the number living at age five to be 69,916, the same as in accompanying tables, then out of 100,821 born alive there will be living at ages (1) 83,143,
(3) 73,684,
(5) 69,916.

This table, joined with the interpolated for ages greater than five, gives for average future duration of life 36.51 years from birth, instead of 36.66, according to the values previously obtained, and adopted in the interpolated and other tables.

If we substitute 562,394, the average annual number *returned* as born alive in Prussia for a period of time (1839, '40, '41) of which the year 1840 was the middle, for 594,851, the approximate number just computed, we shall find

$$\lambda\, p_{0/1} = \bar{1}.9109082 = \lambda\, .81453,$$
$$\lambda\, p_{0/3} = \bar{1}.8544920 = \lambda\, .71531,$$
$$\lambda\, p_{0/5} = \bar{1}.8298000 = \lambda\, .67577\,;$$

and out of 103,461 born alive 84,272 will survive one year,

74,006 " three years,
69,916 " five years,

and the average future duration of life from birth will appear to have been 35.61 years.

A *general interpolation* of the logarithms of the proportions surviving each anniversary of birth intermediate the specified ages, gives the following.

PRUSSIAN LIFE-TABLE, CALCULATED FROM THE AGES OF THOSE DYING DURING THE THREE YEARS 1839, '40, '41; AND FROM THE AGES OF THE LIVING COMPUTED WITH REFERENCE TO THE MIDDLE OF THE YEAR 1840.

Ages.	LOGARITHMS of the Numbers Born, and Living at each Age. λL_x	LOGARITHMS of the Probability, at each Age, of Living One Year. $\lambda L_{x+1} - \lambda L_x$ / λp_x	Differences between Consecutive Logarithms of the Probability of Living. $\lambda p_x - \lambda p_{x+1}$ / $-\Delta \lambda p_x$	PERSONS Born, and Living at each Age. L_x	PERSONS Dying during each Year of Age. $L_x - L_{x+1}$ / D_x	Average Future Duration (or Expectation) of Life. E_x
0	5.001688	.917080 — 1.	— 51797	100,389	17,448	36.66
1	4.918768	.968877	— 10576	82,941	5,736	
2	4.887645	.979453	— 7597	77,205	3,568	
3	4.867098	.987050	— 3378	73,637	2,163	
4	4.854148	.990428	— 1854	71,474	1,558	
5	4.844576	.992282	— 1678	69,916	1,232	47.06
6	4.836858	.993960	— 1241	68,684	948	
7	4.830818	.995201	— 886	67,736	745	
8	4.826019	.996087	— 601	66,991	601	
9	4.822106	.996688	— 381	66,390	504	
10	4.818794	.997069	— 210	65,886	443	44.81
11	4.815863	.997279	— 84	65,443	409	
12	4.813142	.997363	4	65,034	393	
13	4.810505	.997359	63	64,641	392	
14	4.807864	.997296	99	64,249	399	
15	4.805160	.997197	120	63,850	411	41.17
16	4.802357	.997077	121	63,439	425	
17	4.799434	.996956	123	63,014	440	
18	4.796390	.996833	114	62,574	455	
19	4.793223	.996719	107	62,119	468	
20	4.789942	.996612	101	61,651	479	37.54
21	4.786554	.996511	95	61,172	489	
22	4.783065	.996416	95	60,683	499	
23	4.779481	.996321	100	60,184	508	
24	4.775802	.996221	106	59,676	517	
25	4.772023	.996115	118	59,159	527	34.02
26	4.768138	.995997	121	58,632	538	
27	4.764135	.995876	125	58,094	549	
28	4.760011	.995751	128	57,545	560	
29	4.755762	.995623	133	56,985	571	
30	4.751385	.995490	137	56,414	583	30.55
31	4.746875	.995353	140	55,831	594	
32	4.742228	.995213	144	55,237	606	
33	4.737441	.995069	151	54,631	617	
34	4.732510	.994918	153	54,014	628	
35	4.727428	.994765	158	53,386	640	27.14
36	4.722193	.994607	159	52,746	651	
37	4.716800	.994448	161	52,095	661	
38	4.711248	.994287	165	51,434	673	
39	4.705535	.994122	171	50,761	682	
40	4.699657	.993951	185	50,079	693	23.76
41	4.693608	.993766	204	49,386	703	
42	4.687374	.993562	230	48,683	717	
43	4.680936	.993332	258	47,966	731	
44	4.674268	.993074	291	47,235	747	
45	4.667342	.992783	323	46,488	766	20.40
46	4.660125	.992460	352	45,722	787	
47	4.652585	.992108	391	44,935	809	
48	4.644693	.991717	438	44,126	834	
49	4.636410	.991279	498	43,292	860	
50	4.627689	.990781 — 1.	570	42,432	892	17.11

Ages.	Logarithms: Of the Numbers Born, and Living at each Age. λL_x	Logarithms: Of the Probability, at each Age, of Living One Year. $\lambda L_{x+1} - \lambda L_x$ / λp_x	Differences between Consecutive Logarithms of the Probability of Living. $\lambda p_x - \lambda p_{x+1}$ / $-\Delta \lambda p_x$	Persons: Born, and Living at each Age. L_x	Persons: Dying during each Year of Age. $L_x - L_{x+1}$ / D_x	Average Future Duration (or Expectation) of Life. E_x
51	4.618470	.990211 — 1.	651	41,540	925	
52	4.608681	.989560	745	40,615	965	
53	4.598241	.988815	851	39,650	1,008	
54	4.587056	.987964	970	38,642	1,057	
55	4.575020	.986994	1101	37,585	1,108	13.98
56	4.562014	.985893	1307	36,477	1,166	
57	4.547907	.984586	1491	35,311	1,231	
58	4.532493	.983095	1657	34,080	1,302	
59	4.515588	.981438	1803	32,778	1,371	
60	4.497026	.979635	1934	31,407	1,439	11.22
61	4.476661	.977701	2042	29,968	1,500	
62	4.454362	.975659	2137	28,468	1,551	
63	4.430021	.973522	2215	26,917	1,592	
64	4.403543	.971307	2275	25,325	1,619	
65	4.374850	.969032	2323	23,706	1,632	9.03
66	4.343882	.966709	2330	22,074	1,629	
67	4.310591	.964379	2337	20,445	1,610	
68	4.274970	.962042	2345	18,835	1,576	
69	4.237012	.959697	2351	17,259	1,530	
70	4.196709	.957346	2360	15,729	1,471	7.36
71	4.154055	.954986	2367	14,258	1,404	
72	4.109041	.952619	2414	12,854	1,328	
73	4.061660	.950205	2828	11,526	1,249	
74	4.011865	.947377	2988	10,277	1,173	
75	3.959242	.944389	3157	9,104	1,094	5.97
76	3.903631	.941232	3338	8,010	1,014	
77	3.844863	.937894	3527	6,996	932	
78	3.782757	.934367	3726	6,064	851	
79	3.717124	.930641	3939	5,213	769	
80	3.647765	.926702	4162	4,444	690	4.80
81	3.574467	.922540	4399	3,754	613	
82	3.497007	.918141	4648	3,141	540	
83	3.415148	.913493	4913	2,601	470	
84	3.328641	.908580	5191	2,131	404	
85	3.237221	.903389	5485	1,727	345	3.82
86	3.140610	.897904	5799	1,382	289	
87	3.038514	.892105	6126	1,093	241	
88	2.930619	.885979	6475	852	196	
89	2.816598	.879504	6842	656	159	
90	2.696102	.872662	7231	497	127	3.02
91	2.568764	.865431	7641	370	98	
92	2.434195	.857790	8076	272	76	
93	2.291985	.849714	8534	196	57	
94	2.141699	.841180	9019	139	43	
95	1.982879	.832161	9530	96	31	
96	1.815040	.822631	10072	65	22	
97	1.637671	.812559	10644	43	15	
98	1.450230	.801915	11248	28	10	
99	1 252145	.790667	11887	18	7	
100	1.042812	.778780	12562	11	4.4	
101	0.821592	.766218	13275	6.6	2.7	
102	0.587810	.752943	14029	3.9	1.7	
103	0.340753	.738914	14826	2.2	1.0	
104	0.079667	.724088 — 1.		1.2	.6	
105	1.803755			.6	.6	

The leading features in the interpolated Life-Table for Prussia are two.

1st. Strict conformity at certain points to values calculated from actual data.

2d. Regularity in the graduation.

It will be observed that the logarithms of the proportions born alive and surviving ages 1, 3, 5, 14, 25, 35, 45, 55, 65, 75, and 85, as calculated from actual data, are identical with those in the interpolated table. More frequent coincidence would fail, for certain intervals of age, to secure the desired regularity.

From these values we find, by inspection, that the logarithms of the reciprocals of the probabilities of surviving equal consecutive intervals of age diminish from birth, until they attain a minimum between ages 5 and 25 (near age 14), then gradually increase for subsequent intervals.

In effecting the interpolation, we sought to arrive only at results that, coinciding at the ages above specified with those derived from the actual data, should represent the logarithms of the reciprocals of the probabilities of surviving consecutive annual intervals of age as diminishing from birth to a minimum at some point between ages 5 and 25, then gradually increasing for subsequent intervals of age; and that the differences between these logarithms should also advance without manifest irregularity, increasing from, at latest, age 25 to extreme old age.

Two distinct functions of interpolation were employed.

1st. The *exponential.*

For $\lambda\, L_x$, write ϕ_x.

$$\phi_x = \phi_a + (\phi_b - \phi_a)\,\frac{q^{x-a} - 1}{q^{b-a} - 1},$$

in which ϕ_a, ϕ_b, ϕ_c are known values of the function ϕ_x, corresponding to ages a, b, and c. q is to be determined. If the terms be *equidistant*, that is, if $c - b = b - a$,

$$q^{b-a} = \frac{\phi_c - \phi_b}{\phi_b - \phi_a},$$

and

$$\phi_x = \phi_a + \frac{(\phi_b - \phi_a)^2}{(\phi_c - \phi_b) - (\phi_b - \phi_a)}\,.\,(q^{x-a} - 1).$$

If the terms be *not* equidistant, the determining of q will involve the solution of quadratic or higher equations.

2d. The *algebraic.*

$$\phi_x = \phi_a \frac{A_x}{A_a} + \phi_b \frac{B_x}{B_b} + \phi_c \frac{C_x}{C_c} + \phi_d \frac{D_x}{D_d} \ldots\ldots + Q\,\Pi_x\,;$$

in which

$$\Pi_x = x - a \,.\, x - b \,.\, x - c \,.\, x - d \,.\, \ldots\ldots,$$

and

$$A_x = \frac{\Pi_x}{x-a},\ B_x = \frac{\Pi_x}{x-b},\ C_x = \frac{\Pi_x}{x-c},\ D_x = \frac{\Pi_x}{x-d},\ \ldots\ldots;$$

Q may be zero, or an arbitrary constant real and finite, or a real function involving only integral powers of the variable, and which cannot cause the term $(Q\,\Pi_x)$ to become infinite or indeterminate for any value of the variable within the limits assigned for interpolation, or between those corresponding to the extreme given values of the function.

When $x = a$, $\frac{A_x}{A_a}$	obviously becomes *unity*, and terms independent of this factor *vanish.*		
b, $\frac{B_x}{B_b}$	"	"	"
c, $\frac{C_x}{C_c}$	"	"	"
d, $\frac{D_x}{D_d}$	"	"	"

Another convenient function for interpolation when three terms are given, but which was not employed in framing the present table, is the general *parabolic.*

$$\phi_x = \phi_a + (\phi_b - \phi_a)\left(\frac{x-a}{b-a}\right)^q,$$

in which

$$q = \frac{\lambda \dfrac{\phi_c - \phi_a}{\phi_b - \phi_a}}{\lambda \dfrac{c-a}{b-a}}.$$

The exponential involves but *three* known values of the function.

The number of known values that may enter into the interpolation by the algebraic is *unlimited;* but, without care, the resulting series will often be quite eccentric.

Given, logarithms of the proportions born alive, and surviving ages

1, 3, 5, 14, 25, 35, 45, 55, 65, 75, and 85; required ($\lambda\ L_x$) the logarithms of the proportions surviving each intermediate anniversary of birth.

By the exponential formula values between ages

25 and 38	were respectively interpolated from known values of the function for ages	25, 35, and 45.
35 and 48		35, 45, and 55.
45 and 58		45, 55, and 65.
55 and 68		55, 65, and 75.

Values from 73 to age 105, inclusive, were interpolated from known values of the function for ages 65, 75, and 85.

By the above it appears that duplicate values were obtained at ages 36 and 37, 46 and 47, 56 and 57.

The results deduced from these interpolations conform strictly to the conditions imposed, except near the joining points of the several series, where appear irregularities in the first and second orders of differences. These manifest irregularities were then corrected for the several intervals, by adding to the result at each of the several ages the corresponding value of Δ_x, derived from the simple algebraic function

$$\Delta_x = \Delta_g \frac{\overline{x-a}\,.\,\overline{x-b}\,.\,\overline{x-c}\,.\,\overline{x-h}}{\overline{g-a}\,.\,\overline{g-b}\,.\,\overline{g-c}\,.\,\overline{g-h}} +$$

$$\Delta_h \frac{\overline{x-a}\,.\,\overline{x-b}\,.\,\overline{x-c}\,.\,\overline{x-g}}{\overline{h-a}\,.\,\overline{h-b}\,.\,\overline{h-c}\,.\,\overline{h-g}}.$$

In applying the correction to ages between 35 and 48, a, b, c, g, and h equalled respectively 35, 45, 55, 36, and 37. Δ_{36} and Δ_{37} were the differences between the duplicate values for ages 36 and 37. The difference was positive when the one of the duplicate values first obtained was the greater.

In correcting between ages 45 and 58, a, b, c, g, and h equal respectively 45, 55, 65, 46, and 47, and Δ_{46} and Δ_{47} were the differences respectively between the $\lambda\ L_{46}$ and $\lambda\ L_{47}$ just corrected, and the corresponding results derived by the exponential formula from values for ages 45, 55, and 65.

By a similar process the correction was made for values between ages 55 and 68.

A slight irregularity still existing in the second differences (the first differences from the logarithms of the probability of living) near

the joining of the series about age 36, another correction was made to the values between ages 37 and 45, viz.:

$$\Delta_x = \Delta_{34} \frac{x-35 \,.\, x-36 \,.\, x-37 \,.\, x-45 \,.\, x-46 \,.\, x-47}{34-35 \,.\, 34-36 \,.\, 34-37 \,.\, 34-45 \,.\, 34-46 \,.\, 34-47}.$$

The values for Δ_{34}, being the difference between the first of the duplicate $\lambda\, L_{34}$ and the corrected second of the duplicates. Δ_x is additive, if the first of the duplicate values for $\lambda\, L_{34}$ is the greater.

The values between ages 67 and 73, inclusive, were computed from the known values at ages 65, 66, 67, and 73, by assuming the third order of differences constant.

$$\lambda\, L_{73} = \lambda\, L_{65} + 8\,\Delta + 28\,\Delta_2 + 56\,\Delta_3.$$

Δ and Δ_2 were derived from the original value for $\lambda\, L_{65}$, and from the corrected values for $\lambda\, L_{66}$ and $\lambda\, L_{67}$. Δ_3 was then readily found, and consequently the values required between ages 67 and 73. A modification of the method here indicated might have been applied with advantage to the correction of irregularities near the points of junction in other parts of the table.

From the given values of $\lambda\, L_x$ for ages 3, 5, 14, 25, 35, together with the values of $\lambda\, L_x$ for ages 26 and 27, computed as above, the unknown values between ages 5 and 26 were interpolated by the algebraic formula

$$\lambda\, L_x = \phi_x = \phi_3 \frac{A_x}{A_3} + \phi_5 \frac{B_x}{B_5} + \phi_{14} \frac{C_x}{C_{14}} + \phi_{25} \frac{D_x}{D_{25}} + \phi_{26} \frac{E_x}{E_{26}} + \phi_{27} \frac{F_x}{F_{27}} + \phi_{35} \frac{G_x}{G_{35}}.$$

The forms of the functions A, B, C, &c. have been previously given.

From $\lambda\, L_1$, $\lambda\, L_3$, and $\lambda\, L_5$ values were deduced by the exponential formula for $\lambda\, L_2$ (= 4.887645) and $\lambda\, L_4$ (= 4.854456).

By the same formula, from $\lambda\, L_3$, $\lambda\, L_5$, and the computed value for $\lambda\, L_7$ was deduced a duplicate value for $\lambda\, L_4$ (= 4.853532). From comparison of the duplicate values for $\lambda\, L_4$, giving to the former double weight, we obtain 4.854148.

We remark that the desired regularity in the graduation, for the greater part of the table, was attained by making identical three or more consecutive values of adjoining series.

It will be observed that the interpolated results represent mortality

diminishing from birth, until attaining a minimum about age 12, then increasing gradually to age 105, the assumed terminating age of the table. Also, that the values in the column of differences headed $-\Delta \lambda p_x$ gradually increase through the greater part of the entire table, diminishing, however, between ages 17 and 22. A curve, to which the intervals of age and corresponding intensities of mortality are co-ordinates, will be concave downwards through the space where these differences diminish, if elsewhere concave upwards. The attainment of regularity at joining points in the order of differences next higher, was deemed unimportant. For the accuracy with which much of the arithmetical computation has been performed, in the preparation of this and certain other tables following, credit is due to Mr. Howard D. Marshall, of Boston.*

Life-Tables, advancing, by regular gradations, from birth to extreme old age, and conforming strictly at convenient intervals to values derived from original data, are uncommon.

The graduation of the older tables was very imperfect. The Carlisle gives the annual rate of mortality at age 20 greater than at 23; at 31, greater than at 34; at 46 greater than at 51; at 88 greater than at 89; and at 91 the same as at 101. Mr. Milne's excellent table for Sweden and Finland, (1801–5,) though less faulty, is still irregular; so also those of De Parcieux, Kersseboom, Finlaison, and others.

The valuable and elaborate English Life-Tables prepared by Dr. Farr, and published in the Reports of the Registrar-General (England), and also the one prepared by a committee of eminent actuaries to represent a law of mortality according to the combined experience of Insurance Companies, as published by Mr. Jenkin Jones, vary the results derived from actual data, to conform to assumed laws. The graduation of the Actuaries' Table is unexceptionable; that of the tables of Dr. Farr nearly so.

The important tables presented by Mr. E. J. Farren, in his instructive treatise entitled, "Life Contingency Tables, Part I.," begin with age 21, and conform strictly at decennial points to values derived from actual data. The function of interpolation adopted by him was the Calculus of Finite Differences, so far as possible; assuming, however, the intensity of mortality to advance by a constant ratio, when, either from paucity of data or other sufficient cause, the Calculus of

* Mr. Marshall has deceased since this paper was prepared, and in press.

Finite Differences was inapplicable. The results attained are entirely regular.

Many writers on this subject have felt it desirable that some simple generic law be discovered, which, by suitable changes in the constants, will *approximate* the specific laws of human mortality indicated by known tables. Among the more philosophical conceptions is the one of Mr. Gompertz (Philosophical Transactions, 1825), that for the greater part of life man momentarily loses "equal proportions of his remaining power to oppose destruction"; and consequently, that the intensity of mortality increases with advancing age by a constant ratio. Mr. Edmonds would have "the force of mortality at all ages" "expressible by the terms of three geometric series, so connected that the last term of one series is the first of the succeeding series." Dr. Farr recognized the principle in framing his English Tables for 1841; treating "the two series of numbers representing the mortality from 15 to 55, and from 55 to 95, as geometrical progressions. The ratios were derived from a comparison of the increase in the mortality at 15 – 20, 25 – 30, 35 – 40, &c.; and the increase at 20 – 25, 30 – 35, 40 – 45, &c.; and the first terms were derived from these ratios, and the sums of the series which they formed."

Mr. Orchard's method, as described by Mr. Gray in the Assurance Magazine, (London,) for July, 1856, was the adoption of "two consecutive series, having constant second differences," to represent the proportions living from age 20 to 80 and from 80 to 96, the terminating age of his table. He wished "to find a *simple algebraical* relation which should passably well represent some of our best tables." The advantage claimed for a table so constituted "is, that it admits, by the application of simple analytical processes, of the independent formation of any of the values which ordinarily require the aid of a formidable array of the results of previous computation." The same paper gives a *single* algebraical function of the second degree proposed by Mr. Babbage, which is said to represent, nearly, the Swedish Table of Mortality.

Other methods have been proposed by mathematicians of established reputation.*

* A valuable contribution to this department of the science of Vital Statistics was read before the American Association, at its late meeting, by President McCay of South Carolina.

B. Discussion of Certain Methods for converting Ratio of Deaths to Population, within given Intervals of Age, into Logarithms of the Probability that one living at the Earlier Age will attain the Later. With Illustrations from English and Prussian data.

In the paper immediately preceding, a method has been indicated for the conversion of mortality into probability of living from a comparison of *three* consecutive ratios, one preceding and another following the specified interval. In the present paper the results so derived will be compared with others obtained from a *single* ratio.

Let m (identical with M in the preceding paper) represent the rate of annual mortality for any interval of age, or the ratio $\left(\frac{D}{P}\right)$ of the number annually dying to the number living in the community within that interval of age.

If the *population* be *stationary*, $m_{a/b}$ (which equals $\frac{D_{a/b}}{P_{a/b}}$), the rate of annual mortality for the interval between ages a and b, will equal

$$\frac{L_a - L_b}{\int_a^b L_x\,dx} = \frac{\int_a^b - d\,L_x}{\int_a^b L_x\,dx}.$$

If also the *deaths* be supposed *uniformly distributed* throughout the interval of age, i. e. $-d\,L_x$, constant, the numerator $\left(\int_a^b - d\,L_x\right)$ will represent the sum of a series of constants, and the denominator $\left(\int_a^b L_x\,dx\right)$ the sum of a series of values progressing by a common difference; hence the value of the fraction will be *independent of the extent of the interval*, and will vary only with the *mean age*.

If for $\frac{b+a}{2}$, the mean age, we substitute z, and assume for k any arbitrary value, $m_{\overline{z-k}/\overline{z+k}}$ will be *constant for all values of the arbitrary*, provided $2\,k$ does not exceed $(b - a)$ the limits of age within which the uniformity of distribution was assumed.

It will follow that $\frac{L_{z-k}}{L_{z+k}}$, the value of the probability that one living at the earlier age, $z - k$, will attain the later, $z + k$, expressed in terms of the known annual rate of mortality $(m_{a/b})$, and of the arbitrary (k), is

$$\frac{1 - k\ m_{a/b}}{1 + k\ m_{a/b}}.$$

Hence the probability of surviving the entire period $(b - a)$ is

$$\frac{1 - \frac{b-a}{2}\, m_{a/b}}{1 + \frac{b-a}{2}\, m_{a/b}};$$

and the probability of surviving the middle year of the period is

$$\frac{1 - \frac{1}{2}\, m_{a/b}}{1 + \frac{1}{2}\, m_{a/b}}.$$

Again, if the *population* be *stationary*, $m_{x/\overline{x-dx}} \,.\, d\,x$ (for which put $m_{d\,x} \,.\, d\,x$), the intensity of mortality at age x, or the rate of momentary mortality at that age, will equal

$$\frac{-d\,L_x}{L_x} = -d\,\lambda_{\varepsilon}\,L_x = -\lambda_{\varepsilon}\,\frac{L_{\overline{x+dx}}}{L_x} = -\lambda_{\varepsilon}\,p_{x/\overline{x+dx}},$$

(for which put $-\lambda_{\varepsilon}\,p_{d\,x}$), the Napierian logarithm, with the algebraic sign changed, of the probability of surviving a moment of time from age x.

Hence $\int_a^b m_{d\,x}\,d\,x$, the integral within the limits of the ages a and b of the intensity of mortality, will equal $-\lambda_{\varepsilon}\,p_{a/b}$, the Napierian logarithm, with its sign changed, of the probability that one living at the earlier age (a) will attain the later (b).

"A rate of mortality" "derived from the integration $-\frac{1}{L}\,d\,L$" has been happily styled the "integral rate of mortality."*

Assuming *deaths uniformly distributed*, $m_{d\,z}$ becomes equal to $m_{a/b}$; that is, the rate of annual mortality at the mean age equals the rate of annual mortality for the entire interval; consequently

$$-\lambda_{\varepsilon}\,p_{d\,z} = m_{a/b} \,.\, d\,z;$$

that is, the Napierian logarithm, with its sign changed, of the probability of surviving a moment of time at the middle of the specified interval, equals the rate of annual mortality for the interval, multiplied by the differential of the mean age.

* Life Contingency Tables, Part I., by E. J. Farren. In the same connection is stated the important proposition, that "whatever progression prevails among the integral rates of mortality at different ages, the same progression will be found to prevail among the logarithms of the probabilities of living, and *vice versa*."

The intensity of mortality at age z (when deaths are uniformly distributed) being the middle term $\left(\frac{-d\,L_z}{L_z}\right)$ of a series of reciprocals of an arithmetical progression, is less by a small proportion than the average value of the terms constituting the series; hence $(m_{a,b})$ the rate of annual mortality for the interval of age $b - a$ is somewhat *less* (in the case of such uniform distribution) than $\left(\int_a^b m_{dz}\,.\,dz\right)$ the integral rate of mortality for the interval, or than its equivalent $(-\lambda\, p_{a,b})$, the Napierian logarithm, with the sign changed, of the probability that one living at the earlier age (a) will attain the later age (b).

To convert the Napierian to the common logarithm, we multiply by μ $(= .4342945)$, the modulus of the common system.

PRUSSIA. 1839, '40, '41.

TABLE COMPARING LOGARITHMS OF PROBABILITIES OF SURVIVING, COMPUTED BY DIFFERENT METHODS.

Ages a, b.	Ratio of Deaths to Population.	Common Logarithm, with changed Sign, of the Probability that one living at the Earlier Age in each Interval will attain the Later.			
	MORTALITY.	INTEGRAL.	APPROXIMATE.		
	m	$-\lambda\, p$ each from *three* consecutive Ratios.	$-\lambda \dfrac{1 - m\,\frac{b-a}{2}}{1 + m\,\frac{b-a}{2}}$	$-(b-a)\,\lambda \dfrac{1-\frac{m}{2}}{1+\frac{m}{2}}$	$-(b-a)\,\mu\,.\,m$
	A	*B*	*C*	*D*	*E*
0 – 5	.0802238	.157112*	.176598	.174297	.174204
5 – 7	.0152056	.013155	.013208	.013208	.013208
7 – 14	.0077790	.023557	.023655	.023649	.023649
14 – 20	.0062978	.016416	.016413	.016411	.016411
20 – 25	.0089397	.019425	.019416	.019412	.019412
25 – 30	.0096939	.021058	.021054	.021050	.021050
30 – 35	.0108317	.023537	.023527	.023521	.023521
35 – 40	.0131780	.028637	.028626	.028616	.028616
40 – 45	.0144675	.031449	.031430	.031416	.031416
45 – 55	.0210345	.092322	.091691	.091355	.091352
55 – 60	.0357042	.077981	.077738	.077539	.077531
60 – 65	.0557995	.122189	.121962	.121199	.121167
65 – 75	.0909134	.415608	.425992	.395105	.394832
75 – 85	.1515098	.722021	.860283	.659260	.657999
85 and upw'ds	.2661784			1.162896	1.155998

* This value was calculated by a process described in the preceding paper, from population under age 5; from deaths for the intervals of age 0 – 1, 1 – 3, and 3 – 5; and from the rate of annual increase of births estimated from registered births for the six years 1836 – 41.

ENGLAND AND WALES.

Table comparing Logarithms of Probabilities of Surviving, computed by different methods.

Deaths (*Seven Years*) 1838 – 44.
Population computed to Middle of 1841.
Ninth Rep. Reg.-Gen., pp. 176, 177.

Ages a, b.	Ratio of Deaths to Population. MORTALITY. m	Common Logarithm, with changed Sign, of the Probability that one living at the Earlier Age in each Interval will attain the Later. INTEGRAL. $-\lambda p$ Duplicate Values, each from *two* consecutive Ratios.	Mean of the Duplicate Values.	APPROXIMATE. $-\lambda \dfrac{1-m\frac{b-a}{2}}{1+m\frac{b-a}{2}}$	$-(b-a)\lambda \dfrac{1-\frac{m}{2}}{1-\frac{m}{2}}$	$-(b-a)\mu . m$
	A	B		C	D	E
0 – 1	.1792379	.077265	.073073*	.078052	.078052	.077842
1 – 2	.0654971	.028133 .028379	.028256	.028455	.028455	.028445
2 – 3	.0351076	.015206 .015235	.015220	.015249	.015249	.015247
3 – 4	.0250056	.010850 .010854	.010852	.010860	.010860	.010860
4 – 5	.0184203	.007995 .007998	.007997	.008000	.008000	.008000
5 – 10	.0091272	.019686 .019789	.019738	.019823	.019820	.019819
10 – 15	.0052572	.011397 .011425	.011411	.011417	.011416	.011416
15 – 25	.0081967	.035723 .035643	.035683	.035618	.035598	.035598
25 – 35	.0098929	.043033 .043045	.043039	.042999	.042966	.042964
35 – 45	.0124582	.054239 .054261	.054250	.054176	.054105	.054105
45 – 55	.0165886	.072341 .072636	.072489	.072209	.072044	.072043
55 – 65	.0295429	.130216 .130452	.130334	.129249	.128313	.128303
65 – 75	.0622301	.283777 .278012	.280895	.279528	.270349	.270262
75 – 85	.1374474	.718169 .649789	.683979	.731961	.597869	.596926
85 – 95	.2842092	1.127822			1.242716	1.234305
95 – +	.4146003				1.827064	1.800586

* This value was derived from the registered births for the eleven years 1839 – 49, and from the registered deaths under one year of age for the ten years 1840 – 49.

In each of the preceding Tables, column A gives rates of annual MORTALITY for the several specified intervals of age, or ratios of the average numbers annually dying in the community within the specified intervals to the numbers living within the same intervals, estimated with reference to the middle of the year or period in which the deaths occurred.

Column B with changed signs gives the common logarithms of the probabilities of surviving the specified intervals, each computed from THREE consecutive ratios in the column of mortality by a process described in the preceding paper. These values, which we designate *integral* values, may be assumed without appreciable error to represent truly the results demanded by actual data, and with them may be compared *approximate* values obtained by simpler processes.

The *approximate* values in the columns C, D, and E were each derived from SINGLE ratios in A.

The values in C were each obtained by first multiplying (m) the annual rate of mortality by $\left(\frac{b-a}{2}\right)$ half the number of years in the interval of age; then finding the logarithm, with changed sign, of the quotient of unity *less* this product divided by unity *plus* this product.

The values in D were each found by multiplying the number of years in the respective interval by the logarithm, with changed sign, of the quotient of unity *less* half the rate of mortality divided by unity *plus* half the rate.

The values in E were each found by multiplying the mortality by .4342945 (μ), the modulus of the common system of logarithms, and by $(b-a)$ the number of years in the respective interval of age.

Whenever the decrements in the Life-Table resulting from the original data are constant, the corresponding result in C represents the logarithm, with changed sign, of the probability of surviving the *entire interval;* that in D represents the product of $(b-a)$ the number of years in the interval, multiplied by the logarithm, with changed sign, of the probability of surviving the *middle year* of the interval; and that in E the product of $\left(\frac{b-a}{dz}\right)$ the number of equal moments in the interval, multiplied by the logarithm, with changed sign, of the probability of surviving a moment of time at the middle of the interval. Whenever the decrements in the Life-Table are *increasing*, the above

results are each *less* than the respective logarithm; and when *decreasing*, *greater*.

The results in E should in all cases be somewhat less than those in D, although generally the approximation is so close that values in E may without appreciable error be substituted for those in C.

The results in D are likewise less than corresponding ones in C.

The results in C are *less* than the integral values in B, whenever the decrements between the proportions surviving at equidistant ages in the Life-Table, derived from actual data, form a series increasing with the age; they are *equal* to them, when the series is uniform, and *greater* than truth when the series diminishes. By reference to the Prussian Table interpolated for annual intervals, we observe that the decrements diminish from birth to age 13; increase thence to age 65; and again diminish to the age terminating the Table.

The process for deducing values in D is identical with that adopted by Dr. Farr,* in briefly calculating approximate Life-Tables. After determining from values so obtained the proportions of the living at certain ages, he assumed that the proportions within the several intervals were series in arithmetical progression.† It is not unusual, in framing Life-Tables from population and mortality statistics, to let $\frac{1 - \frac{1}{2}m}{1 + \frac{1}{2}m}$ equal the probability of surviving the middle year of a given interval, then, assuming some law of relation, to determine values for intermediate ages. Results so deduced will commonly represent the probability of living for a large part of life somewhat greater than truth demands.

C. Process for deducing accurate Average Duration of Life, Present Value of Life-Annuities, and other useful Tables involving Life-Contingencies, from Returns of Population and Deaths, without the Intervention of a General Interpolation.

The logarithms of the proportions surviving at certain ages ($\lambda\, L_x$) are obtained, by successively adding to the logarithm of a number as-

* To this distinguished writer the science of vital statistics is largely indebted for valuable, extensive, and varied contributions.

† Fifth Report Reg.-Gen. Eng., p. 362.

sumed living at birth, or other specified age, the logarithms of the probabilities of surviving subsequent intervals. Processes for accurately and for approximately determining the logarithms of the probabilities of surviving have been indicated in the previous papers.

Average future duration (or expectation) of life (E_x) expressed in years for any age (x) may be obtained by multiplying by the differential of the age ($d\,x$) the integral of the proportions surviving within the limits of the given age and of the greatest tabular age $\left(\int_x^{105} L_x\right)$, and dividing the product by the proportions living at the given age.

That is,

$$E_x = \frac{d\,x \int_x^{105} L_x}{L_x},$$

in which 105 is assumed the greatest tabular age.

A close approximation to this value may be found by dividing ($\sum_x^{106} L_x = L_x + L_{x+1} + \ldots . L_{105}$) the sum of the proportions living at the given age and at each subsequent anniversary by (L_x) the proportions living at the given age, and from the quotient deducting the half of unity; that is,

$$E_x = \frac{\sum_x^{106} L_x}{L_x} - \tfrac{1}{2}, \text{ nearly.}$$

The latter is the more common process.

The formula expressing the value of a life-annuity, or the present value of one dollar payable at the end of each year during the remainder of the life of the annuitant after attaining a given age, is

$$\frac{L_{x+1}\,v + L_{x+2}\,v^2 + \ldots . L_{105}\,v^{105-x}}{L_x};$$

in which v is the present value of one dollar due one year hence at a given rate of interest.

This expression may readily be converted into the well-known symmetrical form

$$\frac{L_{x+1}\,v^{x+1} + L_{x+2}\,v^{x+2} + \ldots . L_{105}\,v^{105}}{L_x\,v^x},$$

which equals

$$\frac{L_x\,v^x + L_{x+1}\,v^{x+1} + L_{x+2}\,v^{x+2} + \ldots . L_{105}\,v^{105}}{L_x\,v^x} - 1,$$

$$= \frac{\sum_x^{106} L_x v^x}{L_x v^x} - 1.$$

Given (L_0, L_1, L_3, &c.) the proportions born, and surviving ages 1, 3, 5, 14, 25, 35, 45, 55, 65, 75, and 85, according to the law of mortality prevailing in Prussia; required corresponding average future duration of life, life-annuities, and premiums annual and single.

In order to determine values for $\int_x^{105} L_x$ and $\sum_x^{106} L_x v^x$, some law of relation must be supposed to exist between the known values of each of the functions L_x and $L_x v^x$; and this law obviously should represent numbers diminishing with advancing age.

The law may either be expressed by a single formula (as, for instance, the algebraic of the eleventh order), or by a series of distinct formulæ. In consequence of the very great arithmetical labor involved in its practical application, it will not often be thought desirable to adopt a single formula.

When the known values are *equidistant*, n being the number of years in each interval, let

$$S_x = L_x + L_{x+n} + L_{x+2n} + \ldots,$$

and

$$S'_x = L_x v^x + L_{x+n} v^{x+n} + L_{x+2n} v^{x+2n} + \ldots;$$

then will

$$\int_x^{105+} L_x = \int_x^{x+n} S_x,$$

and

$$\sum_x^{106} L_x v^x = \sum_x^{x+n} S'_x.$$

Formulæ which express laws of relation supposed to exist between *four* given values we style *four-point formulæ;* and so for any other number of given values.

The solution of the *four-point algebraic equation*

$$X = \alpha + \beta x + \gamma x^2 + \delta x^3,$$

in which α, β, γ, δ are unknown, and independent of the variable (x), may assume several forms; one of the more convenient of which for our present purpose is

$$X = B\frac{x-c}{b-c} + C\frac{x-b}{c-b} + x-b\,.\,x-c \left\{ \begin{array}{r} \theta_A^2 \dfrac{x-d}{a-d} \\ + \theta_B^2 \dfrac{x-a}{d-a} \end{array} \right\},$$

in which

$$\theta_A^2 = \left\{\frac{C-B}{c-b} - \frac{B-A}{b-a}\right\}\frac{1}{c-a},$$

$$\theta_B^2 = \left\{\frac{D-C}{d-c} - \frac{C-B}{c-b}\right\}\frac{1}{d-b},$$

and A, a, B, b, C, c, D, d, are known corresponding values of the co-ordinates X, x.

Then will

$$d\,x\int_b^c X = \frac{c-b}{2}\left\{C + B - \frac{\overline{c-b}^2}{3}\left\{\begin{array}{l} \theta_A^2\,\dfrac{d-\frac{1}{2}\,\overline{b+c}}{d-a} \\ +\,\theta_B^2\,\dfrac{a-\frac{1}{2}\,\overline{b+c}}{a-d}\end{array}\right\}\right\}$$

$$= \frac{c-b}{2}\,\{C+B+H\};$$

in which H is substituted for

$$-\frac{\overline{c-b}^2}{3}\left\{\begin{array}{l} \theta_A^2\,\dfrac{d-\frac{1}{2}\,\overline{b-c}}{d-a} \\ +\,\theta_B^2\,\dfrac{a-\frac{1}{2}\,\overline{b-c}}{a-d}\end{array}\right\}.$$

Also,

$$\frac{C-B}{2} + \sum_b^c X = \frac{c-b}{2}\left\{C+B+\left(1-\frac{1}{\overline{c-b}^2}\right)H\right\}.$$

If the terms be equidistant, that is, if

$$d-c = c-b = b-a,$$

H becomes

$$\frac{C+B-\overline{D+A}}{12}.$$

Then

$$d\,x\int_b^c X = \frac{c-b}{2}\left\{C+B+\frac{C+B-\overline{D+A}}{12}\right\},$$

and

$$\frac{C-B}{2} + \sum_b^c X = \frac{c-b}{2}\left\{C+B+\frac{C+B-\overline{D+A}}{12}\left(1-\frac{1}{\overline{c-b}^2}\right)\right\}.$$

The *three-point exponential* formula

$$X = \alpha + \beta\,\gamma^x,$$

α, β, and γ being unknown, and independent of the variable x, may take the form

$$X = A + (B - A)\,\frac{q^{x-a} - 1}{q^{b-a} - 1}\,;$$

in which

$$\frac{q^{c-a} - 1}{q^{b-a} - 1} = \frac{C - A}{B - A}\,,$$

whence may be determined the value of q.

$$\left.\begin{array}{r} dx\int_a^b X = \dfrac{1}{Q} \\ \dfrac{B - A}{2} + \sum_a^b X = \tfrac{1}{2}\,\dfrac{q + 1}{q - 1} \end{array}\right\}\ \overline{B - A} + \overline{b - a}\left\{A - \frac{B - A}{q^{b-a} - 1}\right\}$$

$$\left.\begin{array}{r} dx\int_b^c X = \dfrac{1}{Q} \\ \dfrac{C - B}{2} + \sum_b^c X = \tfrac{1}{2}\,\dfrac{q + 1}{q - 1} \end{array}\right\}\ \overline{C - B} + \overline{c - b}\left\{A - \frac{B - A}{q^{b-a} - 1}\right\}$$

Q is the Napierian logarithm of q.

When the terms are equidistant, i. e. $c - b = b - a$,

$$q = \left(\frac{C - B}{B - A}\right)^{\frac{1}{b-a}}\,;$$

$$\frac{1}{Q} = \frac{\overline{b - a}\,.\,\mu}{\lambda\,\dfrac{C - B}{B - A}}\,;$$

and

$$\frac{B - A}{q^{b-a} - 1} = \frac{\overline{B - A}^{2}}{C - B - \overline{B - A}}\,.$$

μ is (.4342945) the modulus of the common system of logarithms.

When the terms are not equidistant, the application of the exponential function involves the resolution of equations of higher than the first degree.

The *three-point parabolic* formula

$$X = \alpha + \beta\,(x - a)^{\gamma}$$

may become

$$X = A + (B - A)\left(\frac{x - a}{b - a}\right)^{q}\,;$$

in which

$$q + 1 = \frac{\lambda\,\dfrac{\overline{C - A}\,.\,\overline{c - a}}{\overline{B - A}\,.\,\overline{b - a}}}{\lambda\,\dfrac{c - a}{b - a}}\,.$$

Then will

$$d\,x \int_a^b X = \overline{b-a}\,A + \frac{\overline{b-a}\,.\,\overline{B-A}}{q+1}$$

and

$$d\,x \int_b^c X = \overline{c-b}\,A + \frac{\overline{c-a}\,.\,\overline{C-A}}{q+1} - \frac{\overline{b-a}\,.\,\overline{B-A}}{q+1}.$$

The *finite* integral of $(x - a)^q$ advances in the form of *a series*, no application of which has been made in the illustrations which follow.

Of the functions above enumerated, the algebraic will commonly prove the most simple in practice, but will not in all cases satisfy the conditions required. When assumed to express the law of relation between certain known values of the function L_x or $L_x v^x$, a portion of the resulting series of numbers between the known values may increase with advancing age, rather than diminish.

The values between B and C, derived from the algebraic formula assigning a law of relation between the *four* known values A, B, C, and D, *lie between* corresponding duplicate values derived from two algebraic formulæ, one a function of the *three* known values A, B, and C, and the other of the *three* values B, C, and D. When the relations of the known values to each other are such that the series resulting from each of the latter formulæ diminish continuously with advancing age from A to C and from B to D respectively, then that portion between B and C of the single algebraic series connecting the four values A, B, C, and D must diminish continuously.

The series of values resulting from the algebraic formula assigning a law of relation between the *three* known values A, B, and C will continuously diminish from A to C *only* when $\frac{A-B}{\overline{b-a}^2}$ and $\frac{B-C}{\overline{c-b}^2}$ are each positive and *greater* than $\frac{A-C}{\overline{c-a}^2}$; or, if the three terms be equidistant, *only* when $\left(\frac{B-A}{C-B}\right)$ the value of the ratio of the first differences is between $\frac{1}{3}$ and 3, and the differences themselves negative. Similar relations obviously obtain when the three known values are B, C, and D.

Applying this test to the Prussian Life-Table, we first find that the algebraic function assigning a law of relation between the *three* known values does not completely satisfy the conditions for the proportions surviving ages 0, 1, 3; 3, 5, 11; 75, 85, 95; and 85, 95, 105; that

is, for the extremes of the table, the values there rapidly diminishing; and also for ages 3, 5, and 14, where there is a great disparity in the length of the intervals of age. It will hereafter appear that eccentricities at the older ages may be disregarded in constructing tables of future duration of life, and of life-annuities, without materially affecting the correctness of the results for earlier ages.

TABLE I.

Average Future Duration of Life in Prussia.

Algebraic Integration.

Ages.	Proportions Born, and Living at Specified Ages in Prussia, calculated by the Integral Method, from *three* Consecutive Ratios of Deaths to Population.			Sum of (L_x) the Proportions Living at the given Age and at all subsequent Ages specified. $L_x+L_{x+10}+L_{x+20}+\ldots$	Aggregate Number of Future Years that (L_x) the Proportions surviving Specified Ages will live. $\frac{10}{2}\left\{S_x+S_{x+10}+\frac{S_x+S_{x+10}-S_{x-10}-S_{x-20}}{12}\right.$	Average Future Duration of Life. $\frac{dx\int_x^{105} L_x}{L_x}$
	L_x	Ages.	L_x	S_x	$dx\int_x^{105} L_x$	E_x
5	69,916	5	69,916	364,916		
14	64,249	15	63,748	295,000	2,626,778	41.21
25	59,159	25	59,159	231,252	2,012,408	34.02
35	53,386	35	53,386	172,093	1,448,721	27.14
45	46,488	45	46,488	118,707	948,047	20.39
55	37,585	55	37,585	72,219	524,773	13.96
65	23,706	65	23,706	34,634	215,941	9.11
75	9,104.2	75	9,104.2	10,927.6	54,597	6.00
85	1,726.7	85	1,726.7	1,823.4	5,847	3.39
		95	96.1	96.7	— 232	
		105	.64	.64		

In the preceding table the *first* of the columns headed L_x gives the proportions surviving certain ages according to the law of mortality prevailing in Prussia; the probability of surviving each of the several intervals being calculated from three consecutive ratios of deaths to populations. In the *second* of the columns headed L_x the values for ages 95 and 105 were computed from the logarithms of the proportions surviving ages 65, 75, and 85, by the exponential formula which expresses the value of the required logarithms in terms of the age, and of the three given logarithms. The number surviving age 15 (63,748) was obtained by assuming an algebraic law of relation for the proportions surviving the *four* ages 5, 14, 25, and 35.

The next column (S_x) gives the sum of the proportions surviving the given age and all subsequent specified ages.

The values in column headed $dx\int_x^{105} L_x$ give the aggregate number of future years of life that the proportions surviving given ages will enjoy, according to the prevailing law of mortality, and were each computed from four equidistant values in the preceding column (S_x), by means of the formula

$$dx\int_x^{105} L_x = \frac{n}{2}\left\{S_x + S_{x+n} + \frac{S_x + S_{x+n} - S_{x-n} - S_{x+2n}}{12}\right\}.$$

The values thus obtained, divided by the corresponding proportions living, give the average future duration of life.

We have already called attention to the unsatisfactory nature of the values resulting from the use of the algebraic formula when the given numbers rapidly diminish, as in the Life-Table after about age 75.

Table II. will compare corresponding results obtained by different formulæ and processes.

In the third column of Table II. the integrations were effected by the exponential formula when the three given values involved in the equation were equidistant; when not equidistant, the parabolic formula was adopted.

The parabolic and the exponential formulæ each afford results that constantly diminish with advancing age.

The process of integration by the algebraic formula involving four known values is the simplest, and between ages 15 to 75 is entirely satisfactory; from 5 to 15, and from 75 upwards, the values afforded are not so reliable; and from 95 to 105, duration of life is represented as *negative.*

In Table IV. we observe that the first three columns of average future duration of life present results, for the larger part of life, *almost identical.*

Values by the algebraic formula slightly exceed those of the following column, calculated by a combination of parabolic, exponential, and algebraic formulæ. The excess at specified ages from 15 to 55 inclusive is only the one-hundredth part (.01) of a year, or *about four days.*

TABLE II.

Comparison of Temporary Aggregate Future Duration of Life, calculated by different Methods, from the Proportions Surviving according to the Prussian Life-Table.

Ages.	Proportions Born alive, and Surviving certain Ages.	The Aggregate Number of Years of Life which the Proportions Surviving at the Commencement of certain Intervals of Age will enjoy during each Interval.				
x	L_x	$dx\int_x^{x+n} L_x$			$\frac{L_{x+n}-L_x}{2}+\Sigma_x^{x+n} L_x$	$\frac{L_{x+n}+L_x}{2}$
		Parabolic and Exponential		Algebraic Formula.	By Annual Interpolation.	Equidifferent Method.
		Duplicates.	Mean.			
0	100,389	87,827*			91,665	91,665
1	82,941	155,560* 155,176			155,494	156,578
3	73,637	142,992 142,359*		142,792	143,251	143,553
5	69,916	664,403* 666,802	665,602	656,124	661,937	668,320
15	63,748	613,405 615,410	614,407	614,370	616,016	614,535
25	59,159	563,826 563,581	563,653	563,687	563,655	562,725
35	53,386	500,393 500,836	500,614	500,674	500,322	499,370
45	46,488	422,257 423,648	422,952	423,274	422,990	420,365
55	37,585	311,573 306,973	309,273	308,830	311,376	306,455
65	23,706	164,599 155,807	160,203	161,341	159,662	164,050
75	9,104	49,992 45,211	47,601	48,750	47,769	54,155
85	1,727	7,137 5,688	6,412	6,081	6,368	9,115
95	96	285 209	247	— 194	228	485
105	1					
115	0					

A comparison of the results in the third column of Table IV. with those arrived at by a general interpolation and direct summation of the proportions surviving each anniversary of birth, exhibits a differ-

* These four values were calculated by the parabolic formula; the other values in the column by the exponential.

ence at specified ages from birth to age 85 inclusive, that in but one case (at age 65) exceeds three one-hundredth parts (.03) of one year, or about *eleven days.*

The former results are deemed in every respect as satisfactory as the latter.

We observe that results by the equidifferent method compared with approved results, from birth to age 45 inclusive, are usually about one tenth of a year in excess; and that for ages above 45 the excess is much greater.

TABLE III.

Future Duration of Life in Prussia.

The Temporary Future Duration of Life for the Proportions Surviving, was computed by the Parabolic Formula from Birth to Age 1 ; *Exponential, from* 1 *to* 3 *and* 3 *to* 5 ; *Mean of Parabolic and Exponential, from* 5 *to* 15 ; *Algebraic, from* 15 *to* 75 ; *and Mean of Exponential Duplicates, from* 75 *to* 105.

Ages.	Temporary Future Duration of Life.	Future Duration of Life.	Average Future Duration of Life.
x	$dx\int_x^{x+n} L_x$	$dx\int_x^{105} L_x$	$\frac{dx\int_x^{105} L_x}{L_x} = E_x$
0	87,827	3,678,033	36.64
1	155,176	3,590,206	43.29
3	142,992	3,435,030	46.66
5	665,602	3,292,038	47.09
15	614,370	2,626,436	41.20
25	563,687	2,012,066	34.01
35	500,674	1,448,379	27.13
45	423,274	947,705	20.39
55	308,830	524,431	13.95
65	161,341	215,601	9.09
75	47,601	54,260	5.96
85	6,412	6,659	3.85
95	247	247	2.57

TABLE IV.

Comparison of Average Future Duration of Life.

Computed, by different Processes, from the Prussian Life-Table.

Ages.	By the Algebraic Formula.	Parabolic, 0 – 1 ; Exponential, 1 - 3 and 3 – 5 : Mean of Parabolic and Exponential, 5 -15: Algebraic, 15 – 75 ; Mean of Exponential Duplicates, 75 – 105	By Annual Interpolation.	Assuming L_x within each Interval to advance by an *Equidifferent Progression.*
—				
0		36.64	36.66	36.77
1		43.29	43.27	43.40
3		46.66	46.63	46.76
5		47.09	47.06	47.19
15	41.21	41.20	41.17	41.28
25	34.02	34.01	34.02	34.09
35	27.14	27.13	27.14	27.24
45	20.39	20 39	20.40	20.53
55	13.96	13.95	13.98	14.21
65	9.11	9.09	9.03	9.61
75	6.00	5.96	5.97	7.00
85	3.39	3.85	3.82	5.55
95		2.57	2.37	5.05

TABLE V.

Value, at certain Ages, of One Dollar to be paid at the End of each Year during the Remainder of Life, according to the Prussian Life-Table, with Process for determining.

Interest of Money, Four per Cent. Algebraic Finite Integration.

Ages.	$L_x\left(\frac{1}{1.04}\right)^x$	$L'_x + L'_{x+10} + L'_{x+20} + \ldots$	$\frac{10}{2}\left\{S'_x + S'_{x+10} + \frac{99}{100}H\right\}$	$\frac{\Sigma_x^{106} L'_x}{L'_x} - 1$
x	L'_x	S'_x	$-\frac{L'_x}{2} + \Sigma_x^{106} L'_x$	Life-Annuity, 4 per Cent.
5	57,466	143,287		
15	35,397	85,821	666,675	18.33
25	22,192	50,424	384,260	16.82
35	13,529	28,232	208,804	14.93
45	7,959	14,703	103,448	12.50
55	4,347	6,744	43,186	9.44
65	1,852	2,397	13,115	6.58
75	481	545	2,306	4.29
85	61.6	63.9	133.6	1.67
95	2.3	2.3	— 13.8	
105	.01	.01		

$$H = S'_x + S'_{x+10} - S'_{x-10} - S'_{x-20}.$$

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In Table V. we observe that the ratios of the first differences of the values $\left(L_x \left(\frac{1}{1.04}\right)^x\right)$ in the second column, for ages 65 and over, are not within the required limits, $\frac{1}{3}$ and 3, and, consequently, that the values of the annuity resulting from integration by the algebraic formulæ are to an extent unsatisfactory.

Had the integration of S'_x for the older ages been effected by the exponential formula, the following would have resulted.

Ages.	$-\frac{L'_x}{2} + \Sigma_x^{x+n} S'_x$		$\frac{\Sigma_x^{x+n} S'_x}{L'_x} - 1$
	Duplicates.	Mean.	
55	43,539.1 / 42,671.2	43,105	9.42
65	13,417.5 / 12,711.6	13,065	6.55
75	2,525.4 / 2,281.3	2,403	4.50
85	233.3 / 188.2	210.8	2.92

In Table VI. the values in the third column corresponding to intervals between ages 65 and 95 are arithmetical means of duplicate values resulting from finite integration of L'_x by the exponential formula. The value opposite age 95 is a *single* value similarly obtained.

Duplicate Values and Mean.

Ages.	Duplicates.	Mean.
65	10,990.8 / 10,356.0	10,673
75	2,312.6 / 2,076.1	2,194
85	229.4 / 183.0	206.2
	Single Value.	
95	6.3	6.3

In the same column (the third) the value of the summation from 5 to 15 (455,694) is the arithmetical mean of a result (456,371) derived by integration from the parabolic formula involving values in the preceding column for ages 3, 5, and 15, and of another (455,017) obtained by finite integration from the exponential formula involving values for ages 5, 15, and 25.

The remaining values in that column were obtained by the finite integration of a series of distinct algebraic formulæ, each involving four given values of $L_x \left(\frac{1}{1.04}\right)^x$.

When the terms in the second column (L'_x) are equidistant, and the intervals *unity*, $1 - \left(\frac{1}{c - b}\right)^2$ in the expression for the finite integral of the algebraic formula becomes *zero*, and the corresponding values in column third become $\frac{L'_x + L'_{x+1}}{2}$.

In the second column ($L_2 v^2$ and $L_4 v^4$) values for ages 2 and 4 were interpolated by means of the exponential formula involving values for ages 1, 3, and 5.

The value of the life-annuity (a_x) at each of the specified ages was obtained by dividing the corresponding value in the fourth column by that in the second, and from the quotient deducting five tenths of unity.

We observe that annuities from age 15 to 45 inclusive, according to Table V., and for subsequent specified ages according to the modification of that table by the exponential formula, are essentially identical with values in Table VI. at corresponding ages.

Our L'_x (constructed according to Barrett's method) corresponds to the D_x of Mr. Griffith Davies and later writers. Our $\sum_x^{106} L'_x$ (or $\sum_x^{x-n} S'_x$) is the N_x employed by Dr. Farr and Mr. Gray, and the N_{x-1} of Mr. Davies, adopted by Mr. David Jones, Mr. Jenkin Jones, Professor De Morgan, and others.

Unaugmented annual and single premiums to insure $ 100, payable at the end of the year of decease, may be computed by the formulæ commonly employed, and heading the respective columns, or be taken directly from Mr. Orchard's very useful tables of "Assurance Premiums."

TABLE VI.

LIFE-ANNUITY, WITH TABLES PREPARATORY; ALSO TABLES OF UNAUGMENTED ANNUAL AND SINGLE PREMIUMS TO INSURE $100, PAYABLE AT THE END OF THE YEAR IN WHICH LIFE SHALL TERMINATE.

Interest of Money, Four per Cent per Annum. Integration by different Formulæ.

Ages.	$L_x\left(\frac{1}{1.04}\right)^x$	$\frac{L'_{x+n} - L'_x}{2} + \Sigma_x^{x+n} L'_x$	$-\frac{L'_x}{2} + \Sigma_x^{106} L'_x$	Life Annuities, 4 per Cent. $\frac{\Sigma_x^{106} L'_x}{L'_x} - 1$	Premiums Unaugmented. Annual. $100\left(\frac{1}{1+a_x} - \frac{.04}{1.04}\right)$	Premiums Unaugmented. Single. $100\left(1 - \frac{.04}{1.04} \cdot \overline{1+a_x}\right)$
x	L'_x			a_x		
0	100,389	90,070	1,478,812	14.23	2.72	41.42
1	79,751	75,565	1,388,743	16.91	1.74	31.12
2	71,380	68,422	1,313,178	17.90	1.45	27.31
3	65,463	63,236	1,244,746	18.51	1.28	24.96
4	61,010	59,238	1,181,520	18.87	1.19	23.58
5	57,466	455,694	1,122,282	19.03	1.15	22.96
15	35,397	282,414	666,588	18.33	1.33	25.65
25	22,192	175,456	384,174	16.82	1.77	31.46
35	13,529	105,356	208,718	14.93	2.43	38.73
45	7,959	60,261	103,362	12.49	3.57	48.12
55	4,347	30,021	43,101	9.42	5.75	59.92
65	1,852	10,673	13,080	6.56	9.38	70.92
75	481	2,194	2,407	4.50	14.34	78.85
85	61.6	206.2	212.5	2.95	21.47	84.81
95	2.3	6.3	6.3			
105	.01					

Methods for determining from the above data the values of other single life benefits, whether uniform, increasing, or decreasing, either for the entire period of life or for limited portions, may readily be devised.

So also methods analogous to those employed in framing the preceding tables may, with advantage, be adopted in constructing tables that shall afford facilities for the ready solution of questions involving *two or more* life contingencies.

We now wish rules for determining, by brief processes, values intermediate between those already obtained.

Either of the formulæ already given may be resorted to; of which the following is the simplest.

$$X = A\,\frac{x-b\,.\,x-c}{a-b\,.\,a-c} + B\,\frac{x-a\,.\,x-c}{b-a\,.\,b-c} + C\,\frac{x-a\,.\,x-b}{c-a\,.\,c-b};$$

A, a, B, b, and C, c being known corresponding values of the co-ordinates X and x.

The algebraic formula involving *four* given values will commonly afford results of a nature entirely satisfactory within the usual limits of inquiry.

Given, A, a, B, b, C, c, and D, d, corresponding known values of X, x; required values intermediate between B and C. If the given terms be equidistant, and

$$n = d - c = c - b = b - a,$$

the algebraic formula will give the following: —

TABLE VII.

SPECIAL FORMULÆ FOR INTERPOLATION, INVOLVING FOUR KNOWN EQUIDISTANT VALUES OF THE FUNCTION.

Algebraic.

Ages.	X
$b+\frac{1}{10}n$	$\frac{1}{10}(9B+C)+\frac{3}{2000}(3\,.\,\overline{9B+C}-19A-11D)$
$+\frac{2}{10}n$	$\frac{1}{10}(8B+2C)+\frac{8}{1000}(8B+2C-6A-4D)$
$+\frac{3}{10}n$	$\frac{1}{10}(7B+3C)+\frac{7}{2000}(3\,.\,\overline{7B+3C}-17A-13D)$
$+\frac{4}{10}n$	$\frac{1}{10}(6B+4C)+\frac{4}{1000}(3\,.\,\overline{6B+4C}-16A-14D)$
$+\frac{5}{10}n$	$\frac{1}{2}(B+C)+\frac{1}{16}(B+C-A-D)$
$+\frac{6}{10}n$	$\frac{1}{10}(4B+6C)+\frac{4}{1000}(3\,.\,\overline{4B+6C}-14A-16D)$
$+\frac{7}{10}n$	$\frac{1}{10}(3B+7C)+\frac{7}{2000}(3\,.\,\overline{3B+7C}-13A-17D)$
$+\frac{8}{10}n$	$\frac{1}{10}(2B+8C)+\frac{8}{1000}(2B+8C-4A-6D)$
$b+\frac{9}{10}n$	$\frac{1}{10}(B+9C)+\frac{3}{2000}(3\,.\,\overline{B+9C}-11A-19D)$

EXAMPLE. — Given, unaugmented annual premiums, from Table VI., corresponding to ages 15, 25, 35, and 45; required the premium for age 28.

The difference between ages 25 and 28 is $\frac{3}{10}$ of (n) the interval of age from 25 to 35; that is 28 is $b + \frac{3}{10}\,n$.

Then

$$X_{28} = \tfrac{1}{10}\,(7\,B + 3\,C) + \tfrac{7}{2000}\,(3\,.\,\overline{7\,B + 3\,C} - 17\,A - 13\,D),$$

in which A, B, C, and D equal respectively 1.33, 1.77, 2.43, and 3.57.

$$\begin{aligned} \tfrac{1}{10}\,(7\,B + 3\,C) &= \ \ 1.968 \\ 3\,(7\,B + 3\,C) &= 59.04 \\ 17\,A + 13\,B &= 69.02\,; \\ \therefore\quad X_{28} &= \ \ 1.93. \end{aligned}$$

Required a value corresponding to age 40, from data in the column headed $-\dfrac{L'_x}{2} + \sum_x^{106} L'_x$. The formula is

$$X_{40} = \tfrac{1}{2}\,(B + C) + \tfrac{1}{16}\,(B + C - A - D),$$

and the given values are for ages 25, 35, 45, and 55.

$$\begin{aligned} B + C &= 312{,}080, \\ A + D &= 427{,}275\,; \\ \therefore\quad X_{40} &= 148{,}840. \end{aligned}$$

Table VII. may take the symmetrical form of

TABLE VIII.

SPECIAL FORMULÆ FOR INTERPOLATION, INVOLVING FOUR KNOWN EQUIDISTANT VALUES OF THE FUNCTION.

Algebraic.

Ages. x	X	
$b + \frac{1}{10}n$	$\frac{1}{10}\,(9\,B + \ \ C)$	$9 \times 1\,(3\,.\,\overline{9\,B + \ \ C} - 19\,A - 11\,D)$
$\frac{2}{10}n$	$\frac{1}{10}\,(8\,B + 2\,C)$	$8 \times 2\,(3\,.\,\overline{8\,B + 2\,C} - 18\,A - 12\,D)$
$\frac{3}{10}n$	$\frac{1}{10}\,(7\,B + 3\,C)$	$7 \times 3\,(3\,.\,\overline{7\,B + 3\,C} - 17\,A - 13\,D)$
$\frac{4}{10}n$	$\frac{1}{10}\,(6\,B + 4\,C)$	$6 \times 4\,(3\,.\,\overline{6\,B + 4\,C} - 16\,A - 14\,D)$
$\frac{5}{10}n$	$\frac{1}{10}\,(5\,B + 5\,C) + \frac{1}{6000}$	$5 \times 5\,(3\,.\,\overline{5\,B + 5\,C} - 15\,A - 15\,D)$
$\frac{6}{10}n$	$\frac{1}{10}\,(4\,B + 6\,C)$	$4 \times 6\,(3\,.\,\overline{4\,B + 6\,C} - 14\,A - 16\,D)$
$\frac{7}{10}n$	$\frac{1}{10}\,(3\,B + 7\,C)$	$3 \times 7\,(3\,.\,\overline{3\,B + 7\,C} - 13\,A - 17\,D)$
$\frac{8}{10}n$	$\frac{1}{10}\,(2\,B + 8\,C)$	$2 \times 8\,(3\,.\,\overline{2\,B + 8\,C} - 12\,A - 18\,D)$
$b + \frac{9}{10}n$	$\frac{1}{10}\,(\ \ B + 9\,C)$	$1 \times 9\,(3\,.\,\overline{\ \ B + 9\,C} - 11\,A - 19\,D)$

And generally, when the four terms A, B, C, and D are equidistant,

$$X = \frac{1}{c-b}(\overline{c-x}\,B + \overline{x-b}\,C) + \frac{1}{6\,(c-b)^3} \times$$

$$\{\overline{c-x}\,.\,\overline{x-b}\,[3\,.\,(\overline{c-x}\,B + \overline{x-b}\,C) - \overline{d-x}\,A - \overline{x-a}\,D]\}.$$

The writer is not aware that any previous attempt has been made to pass by direct and *summary processes* from the immediate results of actual observations to solutions of monetary and other practical questions involving life contingencies, as *accurate* and *reliable* as those obtained by the intervention of a formidable interpolation. In view of the large and rapidly accumulating mass of population and mortality statistics, such processes seem to be demanded.

The *approximate* methods heretofore published have already been adverted to; allusion has also been made to certain formulæ adopted in the construction of *theoretical* Life-Tables, which afford facilities for the independent formation of required monetary and other values.

NOTE.—The average future duration of life for ages 15, 25, 35, 45, and 55, deduced from values in column C, on page 82, are from .1 to .2 of a year *less*, and those derived from values in column E are from .2 to .3 of a year *greater*, than corresponding durations obtained by more accurate methods. Arithmetical means of these results exceed the true values by about .05 of a year. By giving greater comparative weight to values deduced from C, closer approximations will ensue.

[illegible] generally, when the first terms A, B, C, and D are [illegible],

$$[illegible]$$

$$[illegible]$$

The writer is not aware that any previous attempt has been made to pass by direct and [illegible] from the [illegible] of actual observations to [illegible] of [illegible] and other practical [illegible] involving the [illegible] as [illegible] and [illegible] has been obtained by the [illegible] of a [illegible] interpolation. [illegible] of the large and rapidly [illegible] of [illegible] and [illegible], such [illegible] work is [illegible].

The approximate methods [illegible] published [illegible] have been [illegible] has also been [illegible] to [illegible] in the [illegible] of [illegible] for the [illegible] of [illegible] and other values.

Note.—The averages [illegible] on pages 17, 20, 23, and 26 [illegible] column C, on page 32, [illegible] from .1 to .2 of a year [illegible] values in column C are from .2 to .3 of a year greater, [illegible] by more accurate methods. [illegible] of these results exceed the true values by about .05 of a year. By giving greater [illegible] the weights [illegible] closer approximations will result.